CHAPTER - IV.

EFFECT OF LONGITUDINAL INERTIA AND OF SHEAR DEFORMATION ON THE TORSIONAL FREQUENCIES AND NORMAL MODES OF SHORT WIDE-FLANGED THIN-WALLED BEAMS OF OPEN SECTION*.

4.1. INTRODUCTION:

In the analytical studies presented in Chapters II and III, the problems are formulated utilizing the Timoshenko torsion theory (98) and, the effects of longitudinal inertial and shear deformation are neglected assuming the beam to be lengthy compared to the cross sectional dimensions. But the corrections due to longitudinal inertia and shear deformation may be of importance if the effects of cross sectional dimensions on the frequencies of torsional vibration are desired.

Timoshenko torsion theory, though intended to be an improvement over the classical Saint-Venant torsion theory, suffers from the defect that while dispersive in character, very short wavelengths are propagated with infinite velocities. Thus, this improved theory is limited in its description of high-frequency (short-wavelength) vibrations and, because it contains no delay time (infinite velocities), it is not suited for problems involving the response to sharp transients. So ~~much so~~, Timoshenko torsion theory cannot be justified for short wide-flanged beams

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* Results from this Chapter were published by the author, K.V.Apparao and P.K.Sarma in May, 1974 issue of the Journal of the Aeronautical Society of India, see Ref.(49).

and higher modes of vibration.

Though there exists some studies (3,4,70,104) on free torsional vibrations of beams of open section including second order effects such as longitudinal inertia, shear deformation and shear lag, solutions were given only for the simple case of a simply supported beam. Stating that the frequency equations for other boundary conditions are highly transcendental in nature, their solutions were not attempted. The effects of longitudinal inertia and shear deformation on torsional frequencies for various boundary conditions of short wide-flanged thin-walled beams of open section were not yet fully analyzed. Further, it is observed that the torsional frequency values for Indian standard wide-flanged I-beams are not ~~made~~ available until now in the literature. ~~till now.~~

The present chapter therefore deals with exact and approximate analytical solutions of torsional vibrations of short wide-flanged thin-walled beams of open section, for which the shear center and centroid coincide, including the effects of longitudinal inertia and shear deformation. The governing equations of motion are desired using Hamilton's principle. The method of solution used by Huang (69) in the analysis of Timoshenko beam equations in flexural vibrations, is applied to the coupled equations of motion to derive a clear and neat set of frequency and normal mode equations for six common types of simple and finite beams. Solutions are obtained for two complete differential equations in angle of twist and warping angle respectively.

The constants in these solutions are related by any one of the original coupled equations from which the two complete equations are derived. The advantage of this method is that the boundary conditions prescribed are homogeneous and the analysis becomes quite simple. The expressions for orthogonality and normalizing conditions for the principal normal modes, which are useful in solving forced vibration problems and, which include both the angle of twist and warping angle are also obtained in this Chapter for both the general case and for beams with various simple end conditions.

To facilitate use by the designers, extensive design data are presented for the torsional frequencies of Wide-flanged doubly symmetric I-beams with various types of end conditions. The results for the first four modes of vibration for various types of end conditions are presented in tabular form suitable for design use.

To supplement the exact solutions, with approximate analytical solutions, the problem is also solved for some typical boundary conditions utilizing the Galerkin's technique. Depending upon the assumed functions satisfying the prescribed boundary conditions of the beam, Galerkin's technique is found to give nearly accurate results.

4.2. BASIC ASSUMPTIONS:

The problems investigated in this Chapter are restricted to the following assumptions:

a) The material of the beam is homogeneous, isotropic and obeys Hooke's law.

b) By symmetry, the cross sections rotate with respect to centroidal axis, the warping is confined to flanges only.

c) Plane cross sections of different straight pieces remain plane, and warping accross the thickness of these cross sections is neglected.

d) The distortion of the wab out of its plane is assumed negligible.

e) Bending of the flanges does not produce any additional shear stresses on the flange-web section.

f) No internal and external damping forces exist.

g) The deformations are small compared with the cross-sectional dimensions of the beam in the linearized problem.

4.3. DERIVATION OF DIFFERENTIAL EQUATIONS OF MOTION:

Figs.4.1 and 4.2 show a differential element of length dz of a wide-flanged I-beam undergoing torsion. The strain energy U_1 at any instant t in a beam of length L due to Saint-Venant torsion is (See Eq. 2.2a)

$$U_1 = \frac{1}{2}\int_0^L GC_s\left(\frac{\partial\phi}{\partial z}\right)^2 dz \qquad (4.1)$$

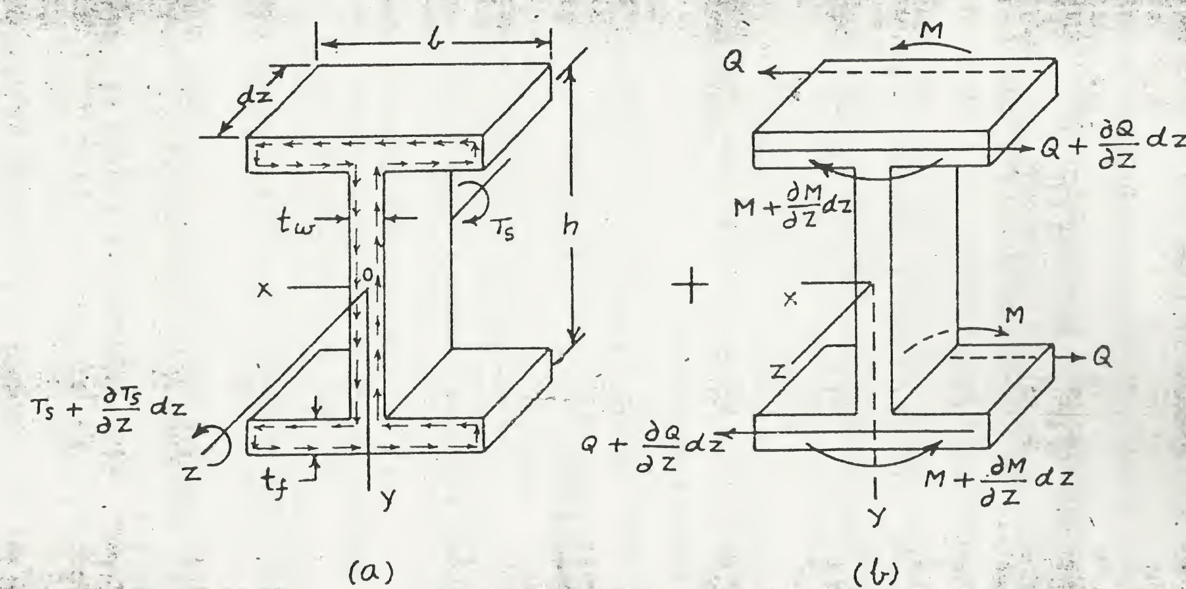

FIG. 4.1 – GEOMETRY AND FORCES ON A DIFFERENTIAL ELEMENT I SECTION

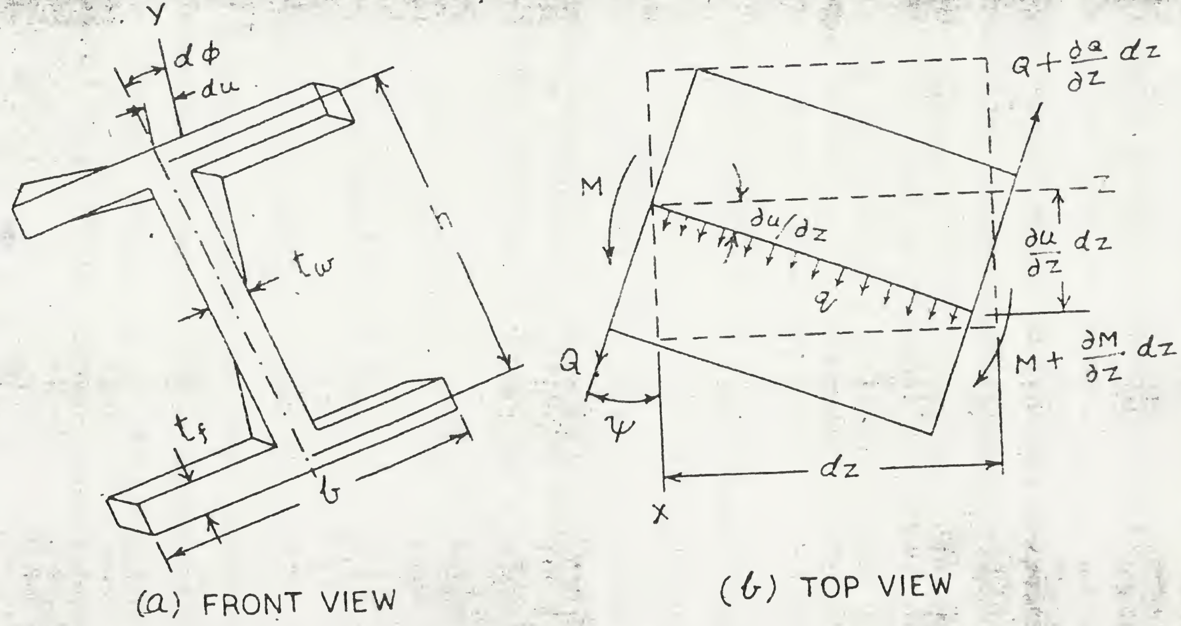

FIG.4.2- STRAINED STATE OF A BEAM ELEMENT

Accompanying the rotation is a warping of the cross-section which is assumed constant in each piece of the cross-section having a moment M. Thus for the wide-flanged section, warping is confined to flanges alone and its angle of rotation denoted by ψ (z,t); see Figs.4.1 and 4.2.

Fig.4.2 (b) shows an element of the top flange. If w is the z-displacement of a point in the top flange, then

$$w = (x, z, t) = - x \psi \qquad (4.2)$$

and the z-component of strain is given by

$$\epsilon_z = \frac{\partial w}{\partial z} = - x \frac{\partial \psi}{\partial z} \qquad (4.3)$$

The section is thin, so we assume $\sigma_x = \sigma_y = 0$, and Hooke's law gives $\sigma_z = E\epsilon_z$, where E is Young's modulus. Moment M due to stresses σ_z is

$$M = EI_f \frac{\partial \psi}{\partial z} \qquad (4.4)$$

It is easily verified that stresses σ_z give rise to no net axial force, and moment M in the top flange and -M in the bottom flange cancel so that no net moment M_y exists on the cross-section. If U_2 is the strain energy of the two flanges due to the warping normal strain (98), then

$$U_2 = \frac{1}{2} \int_0^L 2M(\frac{\partial \psi}{\partial z})dz = \frac{1}{2} \int_0^L 2EI_f(\frac{\partial \psi}{\partial z})^2 dz \qquad (4.5)$$

If ϵ_{sh} is the shear strain at the center of the flange, x = 0, then by definition

$$\epsilon_{sh} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} - \psi \tag{4.6}$$

where u is the x-displacement of the top flange center line. Eq.(4.6) introduces the effect of transverse shear deformation used for bars by Timoshenko (10)) and later applied to plates (7). Using Hooke's law for shear, the value of ϵ_{sh} given by Eq.(4.6) is assumed proportional to the total shear force Q,

$$- Q = K' A_f G \epsilon_{sh} \tag{4.7}$$

where A_f is the cross sectional area of the flange, and K' is the transverse shear coefficient. The equal and opposite shear forces Q, a distance h apart in the top and bottom flanges, give rise to a torque due to warping, T_w, given by

$$T_w = - Qh = K' A_f Gh\left(\frac{h}{2}\frac{\partial \phi}{\partial z} - \psi\right) \tag{4.8}$$

in which displacement compatibility at the web-flange joint

$$u = (h/2)\,\phi \tag{4.9}$$

has been used to eliminate u in Eq.(4.6).

The total torsional couple, T_t, on the cross section is given from Eqs.(2.2a) and (4.8) as

$$T_t = T_s + T_w = GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh\left(\frac{h}{2}\frac{\partial \phi}{\partial z} - \psi\right) \tag{4.10}$$

The strain energy due to shear deformation of the two flanges, U_3, is

$$U_3 = \frac{1}{2}\int_0^L 2(-Q)\,\epsilon_{sh}dz = \frac{1}{2}\int_0^L 2K' A_f G\left(\frac{h}{2}\frac{\partial\phi}{\partial z} - \psi\right)^2 dz \qquad (4.11)$$

The total strain energy, U, at any instant t is given from Eqs.(4.1), (4.5) and (4.11) by

$$U = U_1 + U_2 + U_3 = \frac{1}{2}\int_0^L \left[GC_s\left(\frac{\partial\phi}{\partial z}\right)^2 + 2EI_f\left(\frac{\partial\psi}{\partial z}\right)^2 + 2K' A_f G\left(\frac{h}{2}\frac{\partial\phi}{\partial z} - \psi\right)^2\right]dz \quad (4.12)$$

The total kinetic energy at time t is

$$T_k = \frac{1}{2}\int_0^L \left[\rho I_p\left(\frac{\partial\phi}{\partial t}\right)^2 + 2\rho I_f\left(\frac{\partial\psi}{\partial t}\right)^2\right]dz \qquad (4.13)$$

where the first term is the Kinetic energy of torsional rotation ϕ and the second term is that due to longitudinal (warping) displacements of the two flanges.

Since our object here is to study the free vibrations of the beam, the potential energy, W, of the external force system is taken as zero. If T_k and U from Eqs.(4.12) and (4.13) are substituted into the Hamilton integral given by Eq.(2.1), and variations taken, and after integrating the first two terms by parts with respect to t and next three with respect to z, we obtain:

$$\int_{t_0}^{t_1}\int_0^L \left[\left\{GC_s\frac{\partial^2\phi}{\partial z^2} + K' A_f Gh\left(\frac{h}{2}\frac{\partial^2\phi}{\partial z^2} - \frac{\partial\psi}{\partial z}\right) - \rho I_p\frac{\partial^2\phi}{\partial t^2}\right\}\delta\phi\right.$$

$$\left. + \left\{2EI_f\frac{\partial^2\psi}{\partial z^2} - 2\rho I_f\frac{\partial^2\psi}{\partial t^2} + 2K' A_f G\left(\frac{h}{2}\frac{\partial\phi}{\partial z} - \psi\right)\right\}\bar{\delta}\psi\right]dzdt$$

$$+ \int_0^L \left(\rho I_p\frac{\partial\phi}{\partial t}\delta\phi + 2\rho I_f\frac{\partial}{\partial t}\bar{\delta}\psi\right)\Bigg|_{t_0}^{t_1} dz$$

$$- \int_{t_0}^{t_1} \left[\left\{ GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) \right\} \bar{\delta} \phi + 2EI_f \frac{\partial \psi}{\partial z} \bar{\delta} \psi \right]_0^L dt = 0 \qquad (4.14)$$

Assuming that the values of ϕ and ψ are given at the two fixed instants, the second integral vanishes. If the boundary conditions are such that the third integral also vanishes, then we obtain the following two coupled equations of motion:

$$GC_s \frac{\partial^2 \phi}{\partial z^2} + K' A_f Gh \left(\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z} \right) - \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (4.15)$$

and

$$EI_f \frac{\partial^2 \psi}{\partial z^2} + K' A_f G \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) - \rho I_f \frac{\partial^2 \psi}{\partial t^2} = 0 \qquad (4.16)$$

4.4.(a) NATURAL BOUNDARY CONDITIONS:

In deriving the coupled equations (4.15) and (4.16) from (4.14) it was assumed that the expression

$$\left[GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) \right] \delta \phi + 2EI_f \frac{\partial \psi}{\partial z} \bar{\delta} \psi$$

vanishes at the ends z=0 and z=L. This condition is satisfied if at the two ends,

$$\left[GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) \right] \delta \phi = 0, \qquad (4.17)$$

and

$$\frac{\partial \psi}{\partial z} \bar{\delta} \psi = 0. \qquad (4.18)$$

Eqns.(4.17) and (4.18) give the natural boundary conditions for the finite bar, and are satisfied if the end conditions are taken as:

1. $\phi = 0$ and $\frac{\partial \psi}{\partial z} = 0$ (4.19)

These conditions imply no end rotation and zero bending moment in the flange-ends. In this case, the web is constrained against rotation while the flanges are free to warp. This is the case of a ''Simply Supported end''.

2. $\phi = 0$ and $\psi = 0$ (4.20)

These conditions imply constraint against end rotation as well as end warping, and hence give no end deformation. These conditions define a ''built-in end''.

3. $\frac{\partial \psi}{\partial z} = 0$ and $GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) = 0$ (4.21)

These conditions imply zero bending moment in the flange ends and no torque at the end cross section. The end is thus free from tractions and the conditions correspond to a ''free end''.

4. $\psi = 0, \quad GC_s \frac{\partial \phi}{\partial z} + K' A_f Gh(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) = 0$

or equivalently,

$$\psi = 0, \quad \frac{\partial \phi}{\partial z} = 0 \qquad (4.22)$$

The latter conditions imply no warping and zero shear forces in the end flanges.

These conditions are useful for finding symmetric modes of vibration in simply supported, fixed-fixed, and free-free beams

(b) TIME-DEPENDENT BOUNDARY CONDITIONS:

The homogeneous boundary conditions discussed above give the free vibrations of beams. For forced vibrations produced by the motion of boundaries, appropriate time dependent end conditions are given by prescribing at each end one member of each of the products:

$$\left[GC_s \frac{\partial\phi}{\partial z} + K'A_f Gh\left(\frac{h}{2}\frac{\partial\phi}{\partial z} - \psi\right)\right]\bar{\delta}\phi \text{ and } EI_f \frac{\partial\psi}{\partial z}\bar{\delta}\psi.$$

or equivalently of:

$$T_t\bar{\delta}\phi \text{ and } M\bar{\delta}\psi.$$

Of the many conditions thus obtained, the following are of more theoretical interest;

1. torque T_t prescribed, bending moment $M = 0$ or $\psi = 0$,
2. ϕ or $\frac{\partial\phi}{\partial t}$ prescribed, bending moment $M = 0$ or $\psi = 0$,
3. bending moment M prescribed, torque $T_t = 0$ or $\phi = 0$,
4. ψ or $\frac{\partial\psi}{\partial t}$ prescribed, torque T_t or $\phi = 0$.

In the case of semi-infinite beams, conditions need be prescribed at one end since all physical quantities at any instant are zero at the far end.

4.5.1. SINGLE EQUATION IN ANGLE OF TWIST:

Eliminating ψ between the coupled equations (4.15)and (4.16), a single equation of motion in angle of twist ϕ may be obtained as:

$$\left[\frac{EI_fC_s}{K'A_f} + EC_w\right]\frac{\partial^4\phi}{\partial z^4} - \left[\frac{E\rho I_p I_f}{K'A_fG} + \frac{C_s\rho I_f}{K'A_f} + \frac{\rho I_f h^2}{2}\right]\frac{\partial^4\phi}{\partial z^2\partial t^2}$$

$$- GC_s\frac{\partial^2\phi}{\partial z^2} + \rho I_p\frac{\partial^2\phi}{\partial t^2} + \frac{\rho I_p \rho I_f}{K'A_fG}\frac{\partial^4\phi}{\partial t^4} = 0 \qquad (4.23)$$

Eq.(4.23) is a linear partial differential equation of fourth order, and is of the same form as the Timoshenko beam equation for flexural vibrations (101), under an axial load P which introduces an additional term $- P\frac{\partial^2 y}{\partial z^2}$ (as spring restoring force) in the Timoshenko equation. It is clear that the term $- GC_s\frac{\partial^2\phi}{\partial z^2}$ is analogous to the term $- P\frac{\partial^2 y}{\partial z^2}$.

4.5.2. ANALYSIS OF VARIOUS TERMS:

i) Letting $C_w = \rho I_f = 0$ and $K' \to \infty$, Eq.(4.23) reduces to:

$$GC_s\frac{\partial^2\phi}{\partial z^2} - \rho I_p\frac{\partial^2\phi}{\partial t^2} = 0 \qquad (4.24)$$

This equation represents Saint Venant torsion theory for slender beams and does not include warping of the cross section, shear deformation and longitudinal inertia effects. It is given in Love (76) and is discussed by Gere (32).

ii) $C_w = 0$ and $K' \to \infty$, then Eq.(4.23) becomes:

$$GC_s\frac{\partial^2\phi}{\partial z^2} + \frac{\rho I_f h^2}{2}\frac{\partial^4\phi}{\partial z^2\partial t^2} - \rho I_p\frac{\partial^2\phi}{\partial t^2} = 0 \qquad (4.25)$$

The second term represents Love's corrections(76) for the longitudinal inertia added to Eq.(4.24) and corresponds to Rayleigh's correction(100), for lateral inertia in the elementary theory for longitudinal vibrations.

iii) If $\rho I_f = 0$ and $K' \to \infty$, Eq.(4.23) reduces to:

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (4.26)$$

This equation represents Timoshenko's torsion theory which includes the effect of warping of the cross-section and has been treated in detail by Gere(32).

iv) If $K' \to \infty$, Eq.(4.23) reduces to:

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - \frac{\rho I_f h^2}{2} \frac{\partial^4 \phi}{\partial z^2 \partial t^2} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (4.27)$$

This equation represents Love's correction added to Timoshenko's torsion theory and corresponds to Rayleigh's correction of rotary inertia(100), in the Bernoulli-Euler beam theory.

v) If $\rho I_f = 0$, then Eq.(4.23) is given as:

$$\left(\frac{EI_f C_s}{K' A_f G} + EC_w\right) \frac{\partial^4 \phi}{\partial z^4} - \frac{E \rho I_p I_f}{K' A_f G} \frac{\partial^4 \phi}{\partial z^2 \partial t^2} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (4.28)$$

This equation represents the effect of shear deformation added to Timoshenko's torsion theory.

vi) The part of Eq.(4.23) given by:

$$- \frac{C_s \rho I_f}{K' A_f} \frac{\partial^4 \phi}{\partial z^2 \partial t^2} + \frac{\rho I_p \rho I_f}{K' A_f G} \frac{\partial^4 \phi}{\partial z^4}$$

arises from the coupled interaction of torsional deformation with the bending effects of shear deformation and longitudinal inertia. The $\frac{\partial^4 \phi}{\partial t^4}$ term is responsible for introducing at high frequencies and short wave lengths, a new mode of wave transmission in long bars, and a completely new spectrum of natural frequencies in finite bars.

4.6. NON-DIMENSIONALIZATION AND GENERAL SOLUTION:

Eliminating ϕ in Eqs.(4.15) and (4.16) we obtain the complete differential equation in warping angle ψ as:

$$\left(\frac{EI_f C_s}{K' A_f} + EC_w\right) \frac{\partial^4 \psi}{\partial z^4} - \left(\frac{E \rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right) \frac{\partial^4 \psi}{\partial z^2 \partial t^2}$$

$$- GC_s \frac{\partial^2 \psi}{\partial z^2} + \rho I_p \frac{\partial^2 \psi}{\partial t^2} + \frac{\rho I_p \rho I_f}{K' A_f G} \frac{\partial^4 \psi}{\partial t^4} = 0 \qquad (4.29)$$

Let

$$\phi = \bar{\phi} e^{ip_n t} \qquad (4.30)$$

$$\psi = \bar{\psi} e^{ip_n t} \qquad (4.31)$$

$$Z = z/L \qquad (4.32)$$

where $\bar{\phi}$ is the normal function of ϕ, $\bar{\psi}$ the normal function of ψ, Z the non-dimensional length of beam, $i = \sqrt{-1}$, p_n the natural frequency of vibration.

Substituting Eqs.(4.30) to (4.32) and omitting the factor $e^{ip_n t}$, Eqs.(4.15), (4.16), (4.23) and (4.29) are reduced to:

$$(s^2K^2+1)\bar{\phi}'' + \lambda^2 s^2\bar{\phi} - (2L/h)\bar{\psi}' = 0 \qquad (4.33)$$

$$s^2\bar{\psi}'' - (1-\lambda^2 s^2 d^2)\bar{\psi} + (h/2L)\bar{\phi}' = 0 \qquad (4.34)$$

$$(s^2K^2+1)\bar{\phi}^{iv} + \lambda^2(a^2d^2+s^2)\bar{\phi}'' - \lambda^2(1-\lambda^2 s^2d^2)\bar{\phi} = 0 \qquad (4.35)$$

$$(s^2K^2+1)\bar{\psi}^{iv} + \lambda^2(a^2d^2+s^2)\bar{\psi}'' - \lambda^2(1-\lambda^2 s^2d^2)\bar{\psi} = 0 \qquad (4.36)$$

where

$$a^2 = 1 + s^2K^2 - K^2/\lambda^2 d^2, \qquad (4.37)$$

$$\lambda^2 = \frac{\rho I_p L^4 p_n^2}{EC_w}, \text{ frequency parameter,} \qquad (4.38)$$

$$K^2 = \frac{L^2 GC_s}{EC_w}, \text{ warping parameter,} \qquad (4.39)$$

$$d^2 = \frac{I_f h^2}{2I_p L^2}, \text{ longitudinal inertia parameter,} \qquad (4.40)$$

$$s^2 = \frac{EI_f}{K' A_f G L^2}, \text{ shear deformation parameter} \qquad (4.41)$$

and the primes for $\bar{\phi}$ and $\bar{\psi}$ represent differentiation with respect to Z.

The general solutions of Eqs.(4.35) and (4.36) can be found as:

$$\bar{\phi} = A_1 \cosh \lambda\alpha_2 Z + A_2 \sinh \lambda\alpha_2 Z + A_3 \cos\lambda\beta_2 Z + A_4 \sin\lambda\beta_2 Z \qquad (4.42)$$

$$\bar{\psi} = A_1' \sinh \lambda\alpha_2 Z + A_2' \cosh \lambda\alpha_2 Z + A_3' \sin\lambda\beta_2 Z + A_4' \cos\lambda\beta_2 Z \qquad (4.43)$$

where

$$\begin{matrix}\alpha_2\\ \beta_2\end{matrix} = \frac{1}{\sqrt{2}(s^2K^2+1)^{1/2}}\left\{\mp (a^2d^2+s^2)+\left[(a^2d^2-s^2)^2+4/\lambda^2\right]^{1/2}\right\}^{1/2} \quad (4.44)$$

and

$$\left[(a^2d^2-s^2)^2+4/\lambda^2\right]^{1/2} > (a^2d^2+s^2)$$

is assumed.

In case

$$\left[(a^2d^2-s^2)^2+4/\lambda^2\right]^{1/2} < (a^2d^2+s^2)$$

we write

$$\alpha_2 = \frac{1}{\sqrt{2}(s^2K^2+1)^{1/2}}\left\{(a^2d^2+s^2)-\left[(a^2d^2-s^2)^2+4/\lambda^2\right]^{1/2}\right\}^{1/2}$$

$$= i\,\alpha_2' \quad (4.45)$$

Then Eqs.(4.42) and (4.43) are replaced by

$$\bar{\phi} = A_1\cos\lambda\alpha_2'Z + i\,A_2\sin\lambda\alpha_2'Z + A_3\cos\lambda\beta_2Z + A_4\sin\lambda\beta_2Z \quad (4.46)$$

$$\bar{\psi} = i\,A_1'\sin\lambda\alpha_2'Z + A_2'\cos\lambda\alpha_2'Z + A_3'\sin\lambda\beta_2Z + A_4'\cos\lambda\beta_2Z \quad (4.47)$$

Solutions of Eqs.(4.42) and (4.43) or (4.46) and (4.47) are naturally the solutions of the original coupled equations (4.15) and (4.16).

Only one half of the constants in Eqs.(4.42) and (4.43) are independent. They are related by Eqs.(4.15) or (4.16) as follows:

$$A_1 = \frac{2L}{h\lambda\alpha_2}\left[1 - \lambda^2 s^2(\alpha_2^2 + d^2)\right] A_1' \tag{4.48}$$

$$A_2 = \frac{2L}{h\lambda\alpha_2}\left[1 - \lambda^2 s^2(\alpha_2^2 + d^2)\right] A_2' \tag{4.49}$$

$$A_3 = \frac{2L}{h\lambda\beta_2}\left[1 + \lambda^2 s^2(\beta_2^2 - d^2)\right] A_3' \tag{4.50}$$

$$A_4 = \frac{2L}{h\lambda\beta_2}\left[1 + \lambda^2 s^2(\beta_2^2 - d^2)\right] A_4' \tag{4.51}$$

or

$$A_1' = \frac{h\lambda}{2L}\left[\frac{\alpha_2^2(s^2K^2 + 1) + s^2}{\alpha_2}\right] A_1 \tag{4.52}$$

$$A_2' = \frac{h\lambda}{2L}\left[\frac{\alpha_2^2(s^2K^2 + 1) + s^2}{\alpha_2}\right] A_2 \tag{4.53}$$

$$A_3' = -\frac{h\lambda}{2L}\left[\frac{\beta_2^2(s^2K^2 + 1) - s^2}{\beta_2}\right] A_3 \tag{4.54}$$

$$A_4' = \frac{h\lambda}{2L}\left[\frac{\beta_2^2(s^2K^2 + 1) - s^2}{\beta_2}\right] A_4 \tag{4.55}$$

4.7. FREQUENCY EQUATIONS AND MODAL FUNCTIONS:

In section 4.4(a), natural boundary conditions were discussed. By combining these conditions in pairs, many types of single-span beams can be analyzed. In terms of non-dimensional parameters, the boundary conditions can be written as:

1. Simple Support:

$$\bar{\phi} = 0, \; \bar{\psi}' = 0 \tag{4.56}$$

2. Fixed Support:

$$\bar{\phi} = 0, \; \bar{\psi} = 0 \tag{4.57}$$

3. Free End:

$$\bar{\psi}' = 0, \; (s^2K^2 + 1)\bar{\phi}' - (2L/h)\bar{\psi} = 0 \tag{4.58}$$

The application of appropriate boundary conditions (4.56) to (4.58) and, relations of integration constants (4.48) to (4.55), to equations (4.42) and (4.43) yields for each type of beam a set of four constants A_1 to A_4 with or without primes. In order that the solutions other than zero may exist the determinant of the coefficients of A's must be equal to zero. This leads to the frequency equations in each case and the roots of these frequency equation, λ_i, $i = 1,2,3,\ldots n$, give the eigen values of the problem. The corresponding modal functions, $\bar{\phi}_i$ and $\bar{\psi}_i$, can be obtained accordingly.

4.7.1. SIMPLY SUPPORTED BEAM:

The boundary conditions for a beam simply supported at both ends are:

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at } Z = 0$$

and

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at } Z = 1$$

For the boundary conditions at Z = 0, Eqs.(4.42) and (4.43) give:

$$A_1 + A_3 = 0,$$

$$[\alpha_2^2(s^2K^2+1) + s^2]A_1 - [\beta_2^2(s^2K^2+1) - s^2]A_3 = 0$$

Since the secular determinant, ie., $(s^2K^2+1)(\alpha_2^2+\beta_2^2) \neq 0$, therefore it follows that: $A_1 = A_3 = 0.$ (4.59)

For the second pair of conditions at Z = 1, Eqs.(4.42) and (4.43) give:

$$A_2 \sinh \lambda\alpha_2 + A_4 \sin \lambda\beta_2 = 0,$$

and

$$\left[\alpha_2^2(s^2K^2+1) + s^2\right]A_2 \sinh \lambda\alpha_2 - \left[\beta_2^2(s^2K^2+1) - s^2\right]A_4 \sin \lambda\beta_2 = 0.$$

.. (4.60)

For a non-trivial solution, the secular determinant must vanish. This gives the characterestic equation:

$$(s^2K^2+1)(\alpha_2^2 + \beta_2^2) \sinh\lambda\alpha_2 \sin\lambda\beta_2 = 0 \qquad (4.61)$$

Since $(s^2K^2+1)(\alpha_2^2 + \beta_2^2) \neq 0$, the possible solutions are:

$\lambda\alpha_2 = 0, \quad \lambda\beta_2 = 0;$

$\lambda\alpha_2 = 0, \quad \lambda\beta_2 \neq 0;$

$\lambda\alpha_2 \neq 0, \quad \lambda\beta_2 = 0;$

$\lambda\alpha_2 \neq 0, \quad \lambda\beta_2 = n\pi, \; n=1,2,3,\ldots$

The solution $\lambda\alpha_2=0$, $\lambda\beta_2= 0$ is not valid and the cases $\lambda\alpha_2 \neq 0$, $\lambda\beta_2 = 0$ and $\lambda\alpha_2 = 0$, $\lambda\beta_2 \neq 0$, by Eq.(4.44) imply $\lambda^2= 0$ and

$\lambda^2 = 1/s^2d^2$ respectively. Using the Eqs.(4.42) and (4.43) and following the above procedure for $\lambda^2 = 0$, and for $\lambda^2 = 1/s^2d^2$, we can see that the former case leads to a trivial solution and the latter to:

$$\bar{\phi} = 0, \quad \bar{\psi} = \text{constant} \tag{4.62}$$

The critical frequency $\lambda_c^2 = 1/s^2d^2$ thus represents the first thickness shear mode of the flanges (100). The existence of this mode for the simply supported case of Timoshenko beam in flexural vibrations has been demonstrated by Trail-Nash and Collar (3). It is overlooked by Anderson (3) and neglected by Dolph (3) by a wrong interpretation of the associate results.

The last case:

$$\lambda\alpha_2 \neq 0, \quad \lambda\beta_2 = n\pi, \quad n=1,2,3,\ldots \tag{4.63}$$

leads to the main solution of the problem, Letting $\lambda^2\beta^2 = -n^2\pi^2$ in Eq.(4.44), the frequency equation in λ^2 is obtained as:

$$s^2d^2\lambda^4 - \lambda^2\left[1 + n^2\pi^2(s^2+d^2+s^2d^2K^2)\right] + n^2\pi^2\left[n^2\pi^2(s^2K^2+1) + K^2\right] = 0 \tag{4.64}$$

This equation gives two real positive roots:

$$\lambda_{mn}^2 = \frac{1}{2\,s^2d^2}\left[\left\{1 + n^2\pi^2(s^2+d^2+s^2d^2K^2)\right\} + (-1)^m\left\{\left[1 + n^2\pi^2(s^2-d^2-s^2d^2K^2)\right]^2 + 4n^2\pi^2d^2\right\}^{1/2}\right] \tag{4.65}$$

This frequency equation (4.65) in λ^2, has an infinite number of roots which in general represent two coupled frequency spectra. It may noted that the roots λ^2_{2n} is always $> 1/s^2d^2$. The roots greater than the critical value are also admissible since the same frequency equation is obtained for the case $\lambda^2 > 1/s^2d^2$. Thus, both the roots of (4.65) are admitted and constitute the two uncoupled frequency spectra.

Using (4.63) and (4.60) one gets:

$$A_2 = 0. \qquad (4.66)$$

The modal functions are obtained from Eqs.(4.42) and (4.43) with A's given by (4.59) and (4.66). These are given as:

$$\bar{\phi}_{mn} = \sin n\pi Z \qquad (4.67)$$

$$\bar{\psi}_{mn} = \frac{h}{2n\pi L}\left[n^2\pi^2(s^2K^2+1)-\lambda^2_{mn}\,s^2\right]\cos n\pi Z \qquad (4.68)$$

where λ^2_{mn} being given by (4.65).

The second spectrum appears at higher frequencies, greater than the critical frequency λ_c given by

$$\lambda^2_c = 1/s^2d^2 \qquad (4.69)$$

and is due to interaction between shear deformation and longitudinal inertia. Eq.(4.69) therefore shows the thickness shear nature of the critical frequency while Eq.(4.65) shows the two frequency spectra, uncoupled in the present case.

The classical Timoshenko torsion theory provides only one set of frequency spectrum, while the present analysis provides two frequency spectra. The eigen values λ of the first set of frequency spectrum cover the whole range from zero to infinity, but those of the second set range from the critical frequency λ_c given by equation (4.69) to infinity.

For this case of a simply supported beam, Aggarwal (3), Tso (104) and Krishna Murty and Joga Rao (70) also illustrated two sets of frequency spectra. It is to be mentioned here that for the range of the values of the dimensionless parameters covered in this Chapter, λ is less than λ_c.

For the case, $\lambda > \lambda_c$, it is convenient to use $\alpha_2 = i\alpha_2'$ and, the characterestic frequency equation (4.61) transforms to:

$$\sin \lambda \alpha_2' \sin \lambda \beta_2 = 0 \tag{4.70}$$

where α_2' is given by Eq.(4.45).

Hence, in case there is any extension from there on for λ beyond λ_c ie., $\lambda^2 s^2 d^2 > 1$, care should be taken to account for the frequencies of the second spectrum which can be obtained from Eq.(4.70).

By putting $s^2 = d^2 = 0$ in Eq.(4.64), the equation for the frequency parameter λ, neglecting the effects of shear deformation and longitudinal inertia, can be obtained as:

$$\lambda^2 = n^2\pi^2 (n^2\pi^2 + K^2) \tag{4.71}$$

which is the same as that derived by Gere (32) utilizing Timoshenko torsion theory.

4.7.2. FIXED-FIXED BEAM:

In the case of a beam which is built-in rigidly at both ends, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 0,$$

and

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 1.$$

Applying the above boundary conditions to the general solutions, Eqs.(4.42) and (4.43), the frequency equation, for the first set ($\lambda < \lambda_c$), can be obtained as:

$$2 - 2 \cosh \lambda\alpha_2 \cos \lambda \beta_2$$

$$+ \frac{\lambda\left[\lambda^2 s^2(s^2-a^2d^2)+(3s^2-a^2d^2)\right]}{(1-\lambda^2s^2d^2)^{1/2}(s^2K^2+1)^{1/2}} \sinh \lambda\alpha_2 \sin \lambda\beta_2 = 0 \quad (4.72)$$

The frequency equation for the second set ($\lambda > \lambda_c$) is:

$$2 - 2 \cos \lambda\alpha_2' \cos \lambda \beta_2$$

$$+ \frac{\left[\lambda^2s^2(s^2-a^2d^2)+(3s^2-a^2d^2)\right]}{(\lambda^2s^2d^2-1)^{1/2}\ (s^2K^2+1)^{1/2}} \sin \lambda\alpha_2' \sin \lambda \beta_2 = 0 \quad (4.73)$$

The modal functions for the first set are given by:

$$\bar{\phi} = B(\cosh \lambda\alpha_2 Z + \delta \eta_1 \theta \sinh \lambda \alpha_2 Z - \cos \lambda \beta_2 Z + \eta_1 \sin \lambda \beta_2 Z) \quad (4.74)$$

$$\bar{\psi} = C(\cosh \lambda\alpha_2 Z + \frac{\mu_1}{\delta\theta} \sinh \lambda\alpha_2 Z - \cos \lambda\beta_2 Z + \mu_1 \sin \lambda \beta_2 Z) \quad (4.75)$$

where

$$\delta = \alpha_2/\beta_2$$

$$\theta = \frac{\beta_2^2(s^2K^2+1)-s^2}{\alpha_2^2(s^2K^2+1)+s^2} = \frac{\alpha_2^2(s^2K^2+1)+a^2d^2}{\beta_2^2(s^2K^2+1)-a^2d^2}$$

$$= \frac{\beta_2^2(s^2K^2+1)-s^2}{\beta_2^2(s^2K^2+1)-a^2d^2} = \frac{\alpha_2^2(s^2K^2+1)+a^2d^2}{\alpha_2^2(s^2K^2+1)+s^2}$$

$$\eta_1 = \frac{-\cosh\lambda\alpha_2+\cos\lambda\beta_2}{\delta\theta\sinh\lambda\alpha_2-\sin\lambda\beta_2}$$

$$\mu_1 = \frac{-\cosh\lambda\alpha_2+\cos\lambda\beta_2}{(1/\delta\theta)\sinh\lambda\alpha_2+\sin\lambda\beta_2}$$

The modal functions for the second set are:

$$\bar{\phi} = B(\cos\lambda\alpha_2'Z-\delta'\eta_2\theta\sin\lambda\alpha_2'Z-\cos\lambda\beta_2Z+\eta_2\sin\lambda\beta_2Z)$$

$$\bar{\ } = C(\cos\lambda\alpha_2'Z+\frac{\mu_2}{\delta'\theta}\sin\lambda\alpha_2'Z-\cos\lambda\beta_2Z+\mu_2\sin\lambda\beta_2Z)$$

where

$$\delta' = \alpha_2' / \beta_2$$

$$\eta_2 = \frac{\cos\lambda\alpha_2'-\cos\lambda\beta_2}{\delta'\theta\sin\lambda\alpha_2'-\sin\lambda\beta_2}$$

$$\mu_2 = \frac{-\cos\lambda\alpha_2'+\cos\lambda\beta_2}{(1/\delta'\theta)\sin\lambda\alpha_2'+\sin\lambda\beta_2}$$

Since the coefficients in $\bar{\phi}$ and $\bar{\psi}$ of Eqs.(4.42) and (4.43) are related, the constants B and C, that appear in the modal functions given above are connected through any one of the equations of (4.48) to (4.51) or (4.52)to (4.55).

4.7.3. <u>BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER</u>:

With the end Z = 0, taken as built-in end, and the end Z = 1 as the simply supported end, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 0$$

and

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at} \quad Z = 1.$$

The frequency equation obtained from applying the above boundary condition to the general solutions, Eqs.(4.42) and (4.43), for the first set ($\lambda < \lambda_c$) is given by:

$$\delta\theta \tanh\lambda\alpha_2 - \tan\lambda\beta_2 = 0 \tag{4.85}$$

The frequency equation for the second set ($\lambda > \lambda_c$) is:

$$\delta'\theta \tanh\lambda\alpha_2' + \tan\lambda\beta_2 = 0 \tag{4.86}$$

The modal functions for the first set are given by:

$$\bar{\phi} = B(\cosh\lambda\alpha_2 Z - \coth\lambda\alpha_2 \sinh\lambda\alpha_2 Z - \cos\lambda\beta_2 Z + \cot\lambda\beta_2 \sin\lambda\beta_2 Z) \tag{4.87}$$

$$\bar{\psi} = C(\cosh\lambda\alpha_2 Z + \frac{\mu_3}{\delta\theta} \sinh\lambda\alpha_2 Z - \cos\lambda\beta_2 Z + \mu_3 \sin\lambda\beta_2 Z) \tag{4.88}$$

where

$$\mu_3 = \frac{-(\delta \sinh \lambda\alpha_2 + \sin \lambda\beta_2)}{(1/\theta)\cosh\lambda\alpha_2 + \cos\lambda\beta_2)} \qquad (4.89)$$

The modal functions for the second set are:

$$\bar{\phi} = B(\cos\lambda\alpha_2'Z - \cot\lambda\alpha_2' \sin\lambda\alpha_2'Z - \cos\lambda\beta_2 Z$$

$$+ \cot\lambda\beta_2 \sin\lambda\beta_2 Z) \qquad (4.90)$$

$$\bar{\psi} = C(\cos\lambda\alpha_2'Z - \frac{\eta_3}{\delta'\theta} \sin\lambda\alpha_2'Z - \cos\lambda\beta_2 Z + \eta_3 \sin\lambda\beta_2 Z) \qquad (4.91)$$

where

$$\eta_3 = \frac{\delta' \sin\lambda\alpha_2' - \sin\lambda\beta_2}{(1/\theta)\cos\lambda\alpha_2' + \cos\lambda\beta_2} \qquad (4.92)$$

4.7.4. CANTILEVER BEAM WITH ONE END FIXED AND FREE AT THE OTHER:

For a Cantilever beam built-in rigidly at the end Z=0 so that warping is completely prevented, and with a free end at Z = 1, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 0,$$

and

$$\bar{\ }' = 0, \; (s^2K^2 + 1)\,\bar{\phi}' - (2L/h)\,\bar{\psi} = 0 \quad \text{at} \quad Z = 1.$$

The frequency equation for the first set, in this case, can be obtained as:

$$2 + \left[\lambda^2(a^2d^2 - s^2) + 2\right] \cosh\lambda\alpha_2 \cos\lambda\beta_2$$

$$- \frac{(a^2d^2 + s^2)\,\lambda}{(1-\lambda^2 s^2 d^2)^{1/2}\,(s^2K^2 + 1)^{1/2}} \sinh\lambda\alpha_2 \sin\lambda\beta_2 = 0 \quad (4.\ldots$$

The frequency equation for the second set is given by:

$$2 + \left[\lambda^2(a^2d^2 - s^2) + 2\right]\cos\lambda\alpha_2'\cos\lambda\beta_2$$

$$- \frac{\lambda(a^2d^2 + s^2)}{(\lambda^2 s^2 d^2 - 1)^{1/2}(s^2K^2 + 1)^{1/2}}\sin\lambda\alpha_2'\sin\lambda\beta_2 = 0 \qquad (4.94)$$

The modal functions for the first set are:

$$\bar{\phi} = B(\cosh\lambda\alpha_2 Z - \delta\theta\,\eta_4\sinh\lambda\alpha_2 Z - \cos\lambda\beta_2 Z + \eta_4\sin\lambda\beta_2 Z) \qquad (4.95)$$

$$\bar{\psi} = C(\cosh\lambda\alpha_2 Z + \frac{\mu_4}{\delta\theta}\sinh\lambda\alpha_2 Z - \cos\lambda\beta_2 Z + \mu_4\sin\lambda\beta_2 Z) \qquad (4.96)$$

where

$$\eta_4 = \frac{(1/\delta)\sinh\lambda\alpha_2 - \sin\lambda\beta_2}{\theta\cosh\lambda\alpha_2 + \cos\lambda\beta_2} \qquad (4.97)$$

$$\mu_4 = -\frac{(\delta\sinh\lambda\alpha_2 + \sin\lambda\beta_2)}{(1/\theta)\cosh\lambda\alpha_2 + \cos\lambda\beta_2} \qquad (4.98)$$

The modal functions for the second set are:

$$\bar{\phi} = B(\cos\lambda\alpha_2' Z + \delta'\theta\eta_5\sin\lambda\alpha_2' Z - \cos\lambda\beta_2 Z + \eta_5\sin\lambda\beta_2 Z) \qquad (4.99)$$

$$\bar{\psi} = C(\cos\lambda\alpha_2' Z - \frac{\mu_5}{\delta'\theta}\sin\lambda\alpha_2' Z - \cos\lambda\beta_2 Z + \mu_5\sin\lambda\beta_2 Z) \qquad (4.100)$$

where

$$\eta_5 = \frac{(1/\delta')\sin\lambda\alpha_2' - \sin\lambda\beta_2}{\theta\cos\lambda\alpha_2' + \cos\lambda\beta_2} \qquad (4.101)$$

$$\mu_5 = \frac{\delta'\sin\lambda\alpha_2' - \sin\lambda\beta_2}{(1/\theta)\cos\lambda\alpha_2' + \cos\lambda\beta_2} \qquad (4.102)$$

4.7.5. CANTILEVER BEAM WITH ONE END SIMPLY SUPPORTED AND FREE AT THE OTHER:

For a Cantilever beam simply supported at the end Z=0 and free at Z=1, the boundary conditions are:

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at } Z = 0,$$

and

$$\bar{\psi}' = 0, \ (s^2K^2 + 1)\bar{\phi}' - (2L/h)\bar{\psi} \quad \text{at } Z = 1.$$

The frequency equation for the first set, in this case becomes:

$$\delta \tanh \lambda \alpha_2 - \theta \tan \lambda \beta_2 = 0 \tag{4.103}$$

The frequency equation for the second set is given by:

$$\delta' \tan \lambda \alpha_2' + \theta \tan \lambda \beta_2 = 0 \tag{4.104}$$

The modal functions for the first set are:

$$\bar{\phi} = \frac{\delta \cos \lambda \beta_2}{\cosh \lambda \alpha_2} \sinh \lambda \alpha_2 Z + \sin \lambda \beta_2 Z \tag{4.105}$$

$$\bar{\psi} = \frac{\sin \lambda \beta_2}{\delta \sinh \lambda \alpha_2} \cosh \lambda \alpha_2 Z + \cos \lambda \beta_2 Z \tag{4.106}$$

The modal functions for the second set can be obtained as

$$\bar{\phi} = -\frac{\delta' \cos \lambda \beta_2}{\cos \lambda \alpha_2'} \sin \lambda \alpha_2' Z + \sin \lambda \beta_2 Z \tag{4.107}$$

$$\bar{\psi} = -\frac{\sin \lambda \beta_2}{\delta' \sin \lambda \alpha_2'} \cos \lambda \alpha_2' Z + \cos \lambda \beta_2 Z \tag{4.108}$$

4.7.6. BEAM WITH FREE ENDS:

In the case of a beam which is free at both ends, the boundary conditions are:

$$\bar{\psi}' = 0, \ (s^2K^2+1)\bar{\phi}' - (2L/h)\bar{\psi} = 0 \quad \text{at } Z = 0,$$

and

$$\bar{\psi}' = 0, \ (s^2K^2+1)\bar{\phi}' - (2L/h)\bar{\psi} = 0 \quad \text{at } Z = 1.$$

The frequency equation for the first set, in this case can be obtained as:

$$2 - 2\cosh\lambda\alpha_2 \cos\lambda\beta_2 + \frac{\lambda\left[\lambda^2a^2d^2(a^2d^2-s^2)+(3a^2d^2-s^2)\right]}{(1-\lambda^2s^2d^2)^{1/2}\,(s^2K^2+1)^{1/2}} \sinh\lambda\alpha_2 \sin\lambda\beta_2 = 0 \tag{4.109}$$

The frequency equation for the second set is given by:

$$2 - 2\cos\lambda\alpha_2' \cos\lambda\beta_2 + \frac{\lambda\left[\lambda^2a^2d^2(a^2d^2-s^2)^2+(3a^2d^2-s^2)\right]}{(\lambda^2s^2d^2-1)^{1/2}\,(s^2K^2+1)^{1/2}} \sin\lambda\alpha_2 \sin\lambda\beta_2 = 0 \tag{4.110}$$

The modal functions for the first set can be obtained as:

$$\bar{\phi} = B(\cosh\lambda\alpha_2 Z - \frac{\eta_6}{\delta}\sinh\lambda\alpha_2 Z + \theta\cos\lambda\beta_2 Z + (1/\eta_6)\sin\lambda\beta_2 Z) \tag{4.111}$$

$$\bar{\psi} = C(\cosh\lambda\alpha_2 Z - \frac{\eta_6}{\delta}\sinh\lambda\alpha_2 Z + \theta\cos\lambda\beta_2 Z + (1/\eta_6)\sin\lambda\beta_2 Z) \tag{4.112}$$

where

$$\eta_6 = \frac{\cosh \lambda\alpha_2 - \cos \lambda\beta_2}{\delta \sinh \lambda\alpha_2 - \theta \sin \lambda \beta_2} \quad (4.113)$$

The modal functions for the second set are given by:

$$\bar{\phi} = B(\cos \lambda\alpha_2' Z - \delta' \mu_6 \sin \lambda\alpha_2' Z + (1/\theta)\cos \lambda\beta_2 Z + \mu_6 \sin \lambda \beta_2 Z) \quad (4.114)$$

$$\bar{\psi} = C(\cos \lambda\alpha_2' Z - (\mu_6/\delta')\sin \lambda \alpha_2' Z + \theta \cos \lambda\beta_2 Z + (1/\mu_6)\sin \lambda\beta_2 Z) \quad (4.115)$$

where

$$\mu_6 = \frac{\cos\lambda\alpha_2' - \cos\lambda \beta_2}{\delta' \sin \lambda\alpha_2' + \theta \sin\lambda\beta_2} \quad (4.116)$$

4.8. ORTHOGONALITY AND NORMALIZING CONDITIONS*:

In this section, the expressions for orthogonality and normalizing conditions for the principal normal modes $\bar{\phi}$ and $\bar{\psi}$ are obtained for both the general case and for beams with various simple end conditions.

Let Eq.(4.33) be written in the form

$$\lambda^2 s^2 \bar{\phi} = (2L/h)\bar{\psi}' - (s^2K^2 + 1)\bar{\phi}''$$

for two modes m and n as,

$$\lambda_m^2 s^2 \bar{\phi}_m = (2L/h)\bar{\psi}_m' - (s^2K^2 + 1)\bar{\phi}_m'' \quad (4.117)$$

$$\lambda_n^2 s^2 \bar{\phi}_m = (2L/h)\bar{\psi}_n' - (s^2K^2 + 1)\bar{\phi}_n'' \quad (4.118)$$

* Results from this part of the Chapter were presented by the author and K.V.Apparao at the 16th Congress of ISTAM held at M.N.R.Engineering College, Allahabad, during 29th March to 1st April, 1972. See Ref.(50).

Multiplying Eq.(4.117) by $\bar{\phi}_n$ and Eq.(4.118) by $\bar{\phi}_m$ and subtracting Eq.(4.117) from Eq.(4.118), we have:

$$(\lambda_n^2 - \lambda_m^2)\, s^2 \bar{\phi}_m \bar{\phi}_n = (2L/h)(\bar{\psi}_n' \bar{\phi}_m - \bar{\psi}_m' \bar{\phi}_n) - (s^2K^2+1)(\bar{\phi}_n'' \bar{\phi}_m - \bar{\phi}_m'' \bar{\phi}_n) \tag{4.119}$$

Let Eq.(4.34) be written in the form

$$\lambda^2 s^2 d^2 \bar{\psi} = \bar{\psi} - s^2 \bar{\psi}'' - (h/2L)\,\bar{\phi}'$$

for the two modes m and n as,

$$\lambda_m^2\, s^2 d^2\, \bar{\psi}_m = \bar{\psi}_m - s^2 \bar{\psi}_m'' - (h/2L)\,\bar{\phi}_m' \tag{4.120}$$

$$\lambda_n^2\, s^2 d^2\, \bar{\psi}_n = \bar{\psi}_n - s^2\, \bar{\psi}_n'' - (h/2L)\bar{\phi}_n' \tag{4.121}$$

Multiplying Eq.(4.120) by $\bar{\psi}_n$ and Eq.(4.121) by $\bar{\psi}_m$ and subtracting Eq.(4.120) from (4.121), we get:

$$(\lambda_n^2 - \lambda_m^2)\, s^2 \Omega^2\, \bar{\psi}_m \bar{\psi}_n = (2L/h)(\bar{\phi}_m' \bar{\psi}_n - \bar{\phi}_n' \bar{\psi}_m)$$
$$- (4s^2L^2/h^2)(\bar{\psi}_n'' \bar{\psi}_m - \bar{\psi}_m'' \bar{\psi}_n) \tag{4.122}$$

where

$$\Omega^2 = (4L^2/h^2)d^2 = 2I_f/I_p \tag{4.123}$$

Combining Eqs.(4.119) and (4.122), integrating over the whole beam, and carrying out integration by parts for most of the terms, we obtain:

$$(\lambda_n^2-\lambda_m^2)s^2\int_0^1(\bar{\phi}_m\bar{\phi}_n+\Omega^2\bar{\psi}_m\bar{\psi}_n)\,dZ$$

$$=\int_0^1\Big[(2L/h)(\bar{\psi}_n'\bar{\phi}_m+\bar{\psi}_n\bar{\phi}_m')-(2L/h)(\bar{\psi}_m'\bar{\phi}_n+\bar{\psi}_m\bar{\phi}_n')$$

$$-(s^2K^2+1)(\bar{\phi}_n''\bar{\phi}_m-\bar{\phi}_n\bar{\phi}_m'')-(4s^2L^2/h^2)(\bar{\psi}_n''\bar{\psi}_m-\bar{\psi}_n\bar{\psi}_m'')\Big]dZ$$

$$=\Big[(2L/h)(\bar{\psi}_n\bar{\phi}_m-\bar{\phi}_n\bar{\psi}_m)-(s^2K^2+1)(\bar{\phi}_n'\bar{\phi}_m-\bar{\phi}_n\bar{\phi}_m')$$

$$-(4s^2L^2/h^2)(\bar{\psi}_n'\bar{\psi}_m-\bar{\psi}_n\bar{\psi}_m')\Big]\Big|_0^1 \qquad (4.124)$$

Applying end conditions of any combinations gives the orthogonality condition:

$$\int_0^1(\bar{\phi}_m\bar{\phi}_n+\Omega^2\bar{\psi}_m\bar{\psi}_n)\,dZ=0,\ m\neq n \qquad (4.125)$$

For m = n, the left side of the equations is identically equal to zero because $\lambda_m=\lambda_n$.

Thus the normalizing integral:

$$\int_0^1(\bar{\phi}^2+\Omega^2\bar{\psi}^2)\,dZ$$

cannot be obtained directly by putting m = n in Eq.(4.125)

To evaluate this integral, we let

$$\lambda_m=\lambda \qquad (4.126)$$

$$\lambda_n=\lambda+\bar{\delta}\lambda \qquad (4.127)$$

in which $\bar{\delta}\lambda$ is a small variation of λ, and $\lambda_n=\lambda_m$ as $\bar{\delta}\lambda$ approaches zero. Thus, we have

$$\lambda^2_m = \lambda^2 \tag{4.128}$$

$$\lambda^2_n = (\lambda + \bar{\delta}\lambda)^2 = \lambda^2 + 2\lambda\bar{\delta}\lambda \tag{4.129}$$

in which the higher order small term in the expression of $^{2}_{n}$ is omitted. We also have:

$$\bar{\phi}_n = \bar{\phi}_m + \frac{d\bar{\phi}_m}{d\lambda} \cdot \bar{\delta}\lambda \tag{4.130}$$

$$\bar{\psi}_n = \bar{\psi}_m + \frac{d\bar{\psi}_m}{d\lambda} \cdot \bar{\delta}\lambda \tag{4.131}$$

$$\bar{\phi}'_n = \bar{\phi}'_m + \frac{d\bar{\phi}'_m}{d\lambda} \cdot \bar{\delta}\lambda \tag{4.132}$$

$$\bar{\psi}_n = \bar{\psi}_m + \frac{d\bar{\psi}'_m}{d\lambda} \cdot \bar{\delta}\lambda \tag{4.133}$$

where

$$\frac{d}{d\lambda} = \frac{\partial}{\partial\lambda} + \frac{d\alpha_2}{d\lambda} \cdot \frac{\partial}{\partial\alpha_2} + \frac{d\beta_2}{d\lambda} \cdot \frac{\partial}{\partial\beta_2} \tag{4.134}$$

Substituting the above relations in Eq.(4.124) we obtain:

$$2\lambda\bar{\delta}\lambda s^2 \int_0^1 (\bar{\phi}^2_m + \Omega^2 \bar{\psi}^2_m)\, dZ$$

$$= \Bigg[(2L/h)\Big(\frac{d\bar{\psi}_m}{d\lambda}\bar{\phi}_m - \frac{d\bar{\phi}_m}{d\lambda}\bar{\psi}_m\Big) - (s^2K^2+1)\Big(\frac{d\bar{\phi}'_m}{d\lambda}\bar{\phi}_m - \frac{d\bar{\phi}_m}{d\lambda}\bar{\phi}'_m\Big)$$

$$- (4s^2L^2/h^2)\Big(\frac{d\bar{\psi}'_m}{d\lambda}\bar{\psi}_m - \frac{d\bar{\psi}_m}{d\lambda}\bar{\psi}'_m\Big)\Bigg]\Bigg|_0^1 \delta\lambda \tag{4.135}$$

Dropping the subscript m, dividing both sides of the equation by

$2\lambda\bar{\delta}\lambda s^2$, and rearranging:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)dZ = \frac{1}{2\lambda s^2}\Bigg\{\bar{\phi}\frac{d}{d\lambda}\Big[(2L/h)\bar{\psi} - (s^2K^2+1)\bar{\phi}'\Big]$$

$$+\Big[(s^2K^2+1)\bar{\phi}' - (\frac{2L}{h})\bar{\psi}\Big]\frac{d\bar{\phi}}{d\lambda} - (\frac{4s^2L^2}{h^2})\Big[\frac{d\bar{\psi}'}{d\lambda}\bar{\psi} - \frac{d\bar{\psi}}{d\lambda}\bar{\psi}'\Big]\Bigg\}\Bigg|_0^1 \delta \quad (4.136)$$

This expression can be further simplified for beams of various end conditions as follows:

(1) Simply Supported beam:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)dZ = \frac{1}{2\lambda^2 s^2}\Bigg\{\Big[(s^2K^2+1)\bar{\phi}' - (\frac{2L}{h})\bar{\psi}\Big]\frac{d\bar{\phi}}{d\lambda}$$

$$+ (\frac{4s^2L^2}{h^2})\bar{\psi}\frac{d\bar{\psi}'}{d\lambda}\Bigg\}\Bigg|_0^1 \quad (4.137)$$

(2) Fixed-End Beam:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)dZ = \frac{1}{2\lambda^2 s^2}\Bigg[(s^2K^2+1)\bar{\phi}'\frac{d\bar{\phi}'}{d\lambda} +$$

$$+ (\frac{4s^2L^2}{h^2})\bar{\psi}'\frac{d\bar{\psi}}{d\lambda}\Bigg]\Bigg|_0^1 \quad (4.138)$$

(3) Beam Free at both ends:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)dZ = \frac{1}{2\lambda^2 s^2}\Bigg\{\bar{\phi}\frac{d}{d\lambda}\Big|(\frac{2L}{h})\bar{\psi} - (s^2K^2+1)\bar{\phi}'\Big]$$

$$- \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi} \; \frac{d\bar{\psi}'}{d\lambda} \Bigg]_0^1 \qquad (4.139)$$

(4) Beam fixed at one end, simply supported at the other:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)\, dZ = \frac{1}{2\lambda^2 s^2} \Bigg[\Bigg\{\Bigg[(s^2K^2+1)\bar{\phi}' - \left(\frac{2L}{h}\right)\bar{\psi}\Bigg] \frac{d\bar{\phi}}{d\lambda}$$

$$+ \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi} \; \frac{d\bar{\psi}'}{d\lambda}\Bigg\}_{Z=1} - \Bigg\{(s^2K^2+1)\bar{\phi}' \; \frac{d\bar{\phi}}{d\lambda} + \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi}' \; \frac{d\bar{\psi}}{d\lambda}\Bigg\}_{Z=0}\Bigg]$$

(4.140)

(5) Cantilever beam fixed at one end, free at the other:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)\, dZ = \frac{1}{2\lambda s^2}\Bigg[\Bigg\{\bar{\phi} \; \frac{d}{d\lambda}\Bigg|\left(\frac{2L}{h}\right)\bar{\psi} - (s^2K^2+1)\,\bar{\phi}'\Bigg]$$

$$- \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi} \; \frac{d\bar{\psi}}{d\lambda}\Bigg\}_{Z=1} - \Bigg\{(s^2K^2+1)\bar{\phi}' \; \frac{d\bar{\phi}}{d\lambda} + \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi}' \; \frac{d\bar{\psi}}{d\lambda}\Bigg\}_{Z=0}\Bigg].$$

(4.141)

(6) Cantilever beam simply supported at one end, free at the other:

$$\int_0^1 (\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)\, dZ = \frac{1}{2\lambda^2 s^2}\Bigg[\;\bar{\phi} \; \frac{d}{d\lambda}\Bigg[\left(\frac{2L}{h}\right)\bar{\psi} - (s^2K^2+1)\,\bar{\phi}'\Bigg]$$

$$- \left(\frac{4s^2L^2}{h^2}\right) \; \frac{d\bar{\psi}'}{d\lambda}\Bigg\}_{Z=1} - \Bigg\{\Bigg[(s^2K^2+1)\bar{\phi}' - \left(\frac{2L}{h}\right)\bar{\psi}\Bigg] \frac{d\bar{\phi}}{d\lambda}$$

$$+ \left(\frac{4s^2L^2}{h^2}\right)\bar{\psi} \; \frac{d\bar{\psi}'}{d\lambda}\Bigg\}_{Z=0}\Bigg] \qquad (4.142)$$

It is also suggested that the normalizing integral can be approximated by discrete values of $\bar{\phi}$ and $\bar{\psi}$ along the beam.

<u>Expression of Normalizing condition</u>:

Let Eqs.(4.33) and (4.34) be written as:

$$\lambda^2 s^2 \bar{\phi} = -(s^2K^2 + 1)\bar{\phi}'' + (2L/h)\bar{\psi}' \quad (4.143)$$

$$\lambda^2 s^2 d^2 \bar{\psi} = -s^2 \bar{\psi}'' + \bar{\psi} - (h/2L)\bar{\phi}' \quad (4.144)$$

Multiplying the Eq.(4.143) by $\bar{\phi}$ and the Eq.(4.144) by $\bar{\psi}$, adding the resulting equations, integrating over the whole beam, and carrying out some integrals by integration by parts, we have:

$$\lambda^2 s^2 \int_0^1 (\bar{\phi}^2 + \Omega^2 \bar{\psi}^2)dZ = \int_0^1 \Big[-(s^2K^2+1)\bar{\phi}\bar{\phi}'' + (\tfrac{2L}{h})(\bar{\phi}\bar{\psi}' - \bar{\phi}'\bar{\psi}')$$

$$+ (\tfrac{4L^2}{h^2})\bar{\psi}^2 - (\tfrac{4s^2L^2}{h^2})\bar{\psi}\,\bar{\psi}'' \Big] dZ$$

$$= \int_0^1 \Big[(s^2K^2+1)\bar{\phi}'^2 - (\tfrac{4L}{h})\bar{\phi}'\bar{\psi} + (\tfrac{4s^2L^2}{h^2})\bar{\psi}'^2 + (\tfrac{4L^2}{h^2})\bar{\psi}^2 \Big] dZ \quad (4.145)$$

Eq.(4.145) is the expression of the Normalizing condition which is very useful in analyzing the forced vibration problems.

4.9. APPROXIMATE SOLUTIONS BY GALERKIN'S TECHNIQUE*:

In this section, approximate solutions are obtained, for the problem of free torsional vibrations of thin-walled beams of open section including the effects of longitudinal inertia and shear deformation, utilizing the well-known Galerkin's technique. Solutions with Galerkin's method are illustrated for fixed-fixed beam and for a beam fixed at one end and simply supported at the other.

4.9.1. FIXED-FIXED BEAM:

To satisfy the above boundary conditions in this case, the normal function $\bar{\phi}$ can be assumed in the form

$$\bar{\phi} = \sum_{n=1}^{\infty} D_n (1 - \cos 2n\pi Z) \qquad (4.146)$$

Substituting Equation (4.146)in the differential Equation (4.35), orthogonalizing the resulting error with the assumed function, integrating the obtained function over the whole length of the beam and equating it to zero, the frequency equation in λ^2 can be obtained as:

$$3\lambda^4 s^2 d^2 - \lambda^2 \left| 3+4n^2\pi^2(s^2+d^2+s^2d^2K^2) \right| + 4n^2\pi^2 \left| 4n^2\pi^2(s^2K^2+1)+K^2 \right| = 0 \qquad (4.147)$$

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* Results from this part of the chapter were presented at the 17th Congress of Indian Society of Theoretical and Applied Mechanics, held at Birla Institute of Technology, Mesra, Ranchi, during December 22-25 1972. Ref (51)

Eq.(4.147) gives two real positive roots given by

$$\lambda^2_{mn} = \frac{1}{6s^2d^2}\left[\left\{3+4n^2\pi^2(s^2+d^2+s^2d^2K^2)\right\} + (-1)^m\left\{\left[3+4n^2\pi^2(s^2+d^2+s^2d^2K^2)\right]^2 - 48\,n^2\pi^2s^2d^2\left[4n^2\pi^2(s^2K^2+1)+K^2\right]\right\}^{1/2}\right] \quad (4.148)$$

In arriving at Eq.(4.148), only one term of the infinite series of Eq.(4.146) is utilized. Hence, Eq.(4.148) gives upper bounds and has an infinite number of roots which in general represent two coupled frequency spectra.

By putting $s^2 = d^2 = 0$, Eq.(4.147) reduces to:

$$3\lambda^2 - 4\,n^2\pi^2(4n^2\pi^2 + K^2) = 0 \quad (4.149)$$

and the expression for the frequency parameter λ becomes:

$$\lambda_n = \frac{2n\pi}{\sqrt{3}}(4n^2\pi^2+K^2)^{1/2} \quad (4.150)$$

which is same as that from Eq.(2.73) for $\Delta^2 = \gamma^2 = 0$.

4.9.2. <u>BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER</u>:

The normal function satisfying the boundary conditions in this case can be assumed in the form:

$$\bar{\phi} = \sum_{n=1}^{\infty} D_n \left(\cos\frac{n\pi}{2}Z - \cos\frac{3n\pi}{2}Z\right) \quad (4.151)$$

Substituting Eq.(4.151) in the Eq.(4.35) and following the Galerkin's method, the frequency equation in λ^2 can be obtained as:

$$16\lambda^4 s^2d^2 - \lambda^2 \left[16+20\, n^2\pi^2(s^2+d^2+s^2d^2K^2)\right]$$

$$+\, n^2\pi^2 \left[41\, n^2\pi^2(s^2K^2+1)+20\, K^2\right] = 0 \qquad (4.152)$$

From Eq.(4.152) we have:

$$\lambda^2_{mn} = \frac{1}{16\, s^2d^2}\Big[\Big\{16+20\, n^2\pi^2(s^2+d^2+s^2d^2K^2)$$

$$+ (-1)^m \Big\{\left[16+20\, n^2\pi^2(s^2+d^2+s^2d^2K^2)\right]^2$$

$$- 64\, n^2\pi^2 s^2d^2\left[41\, n^2\pi^2(s^2K^2+1)+20\, K^2\right]\Big\}^{1/2}\Big] \qquad (4.153)$$

By putting $s^2 = d^2 = 0$, Eq.(4.152) reduces to:

$$16\lambda^2\, n^2\pi^2\,(41\, n^2\pi^2+20\, K^2) = 0 \qquad (4.154)$$

and the expression for the frequency parameter λ becomes:

$$\lambda = \frac{n\pi}{4}\,(41\, n^2\pi^2+20\, K^2)^{1/2} \qquad (4.155)$$

which is same as that from Eq.(2.76) for $\Delta^2 = \gamma^2 = 0$.

4.10. RESULTS AND CONCLUSIONS:

For a given beam with K, s and d known, the λ_i (i=1,2,3,...) can be found from the appropriate frequency equations and the corresponding p_i are then calculated by Eq.(4.38). However, these frequency equations are highly transcendental and cannot ~~not to~~ be solved simply. This difficulty is overcome by the use of bisection method on digital Computer IBM 1130 at the Computer Center, Andhra University, Waltair. The results are obtained for some typical boundary conditions and various combinations of K, s and d. The results are presented for the special case s = 2d, which is usually the case for many Indian Standard wide-flanged I-beams.

Let λ_o be the classical eigen values obtained in Chapter II neglecting the effects of longitudinal inertia and shear deformation and p_o, the natural torsional frequencies corresponding to λ_o. Comparing the mechanism of vibration of the classical beam based on Timoshenko Torsion theory and the present beam based on the improved theory, we note that the classical beam is equivalent to the present beam with longitudinal inertia and shear constraints.

Therefore,

$$p \leqslant p_o$$

and

$$\lambda / \lambda_o = p/p_o = q, \; q < 1$$

The ratio of λ / λ_o or p/p_o, denoted by q, will be referred to the ''modifying quotient''. The variation of the ratio λ/λ_0 (also the modifying quotient q) with the longitudinal inertia parameter d for the first three modes of vibration of a simply supported beam is plotted in Fig.4.3, which shows the corrections in the natural torsional frequencies owing to the individual influence of longitudinal inertia. In plotting this figure the warping parameter is taken as equal to 1.0 and the shear parameter s as equal to zero. It can be observed from Fig.4.3 that the reduction in the torsional frequency due to longitudinal inertia increases with increasing values of d. For a maximum value of d = 0.1, the reduction in the torsional frequency can be observed from the graph as about 10 percent for the first mode, 35 percent for the second mode and 65 percent for the third mode. Therefore it can be concluded that the influence of longitudinal inertia on the torsional frequencies increases profoundly for higher modes of vibration.

For a simply supported beam, its higher harmonic corresponds to the fundamental of another simply supported beam of shorter span. The nth frequency of simply-supported beam of span L is equal to the fundamental of another such beam with span L/n. So, for the sake of simplicity and ease of presentation, Fig.4.4 is plotted between the ratio λ/λ_0 and K/n for values of ns = 0.5, 1.0 and 2.0. For constant values of K and s the values of λ/λ_0 can be read from this figure for different values of n (ie., for different modes of vibration). If n is kept constant, the values of λ/λ_0 can be obtained for various combinations of the warping parameter K and shear parameter s. In plotting

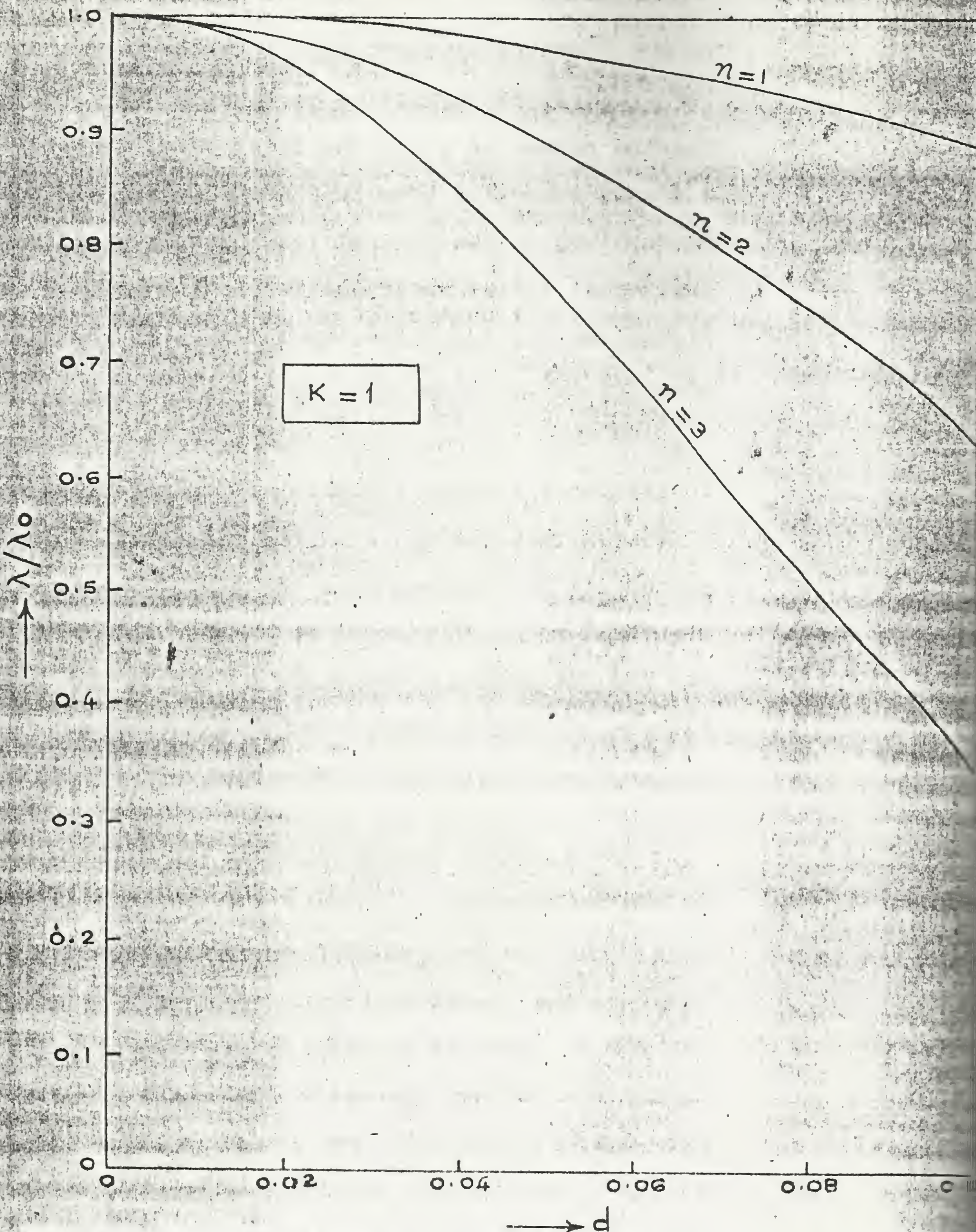

Fig. 4.3. Corrections in natural frequencies of a simply sup-
beam owing to longitudinal inertia for the first three
of Vibration (S=0)

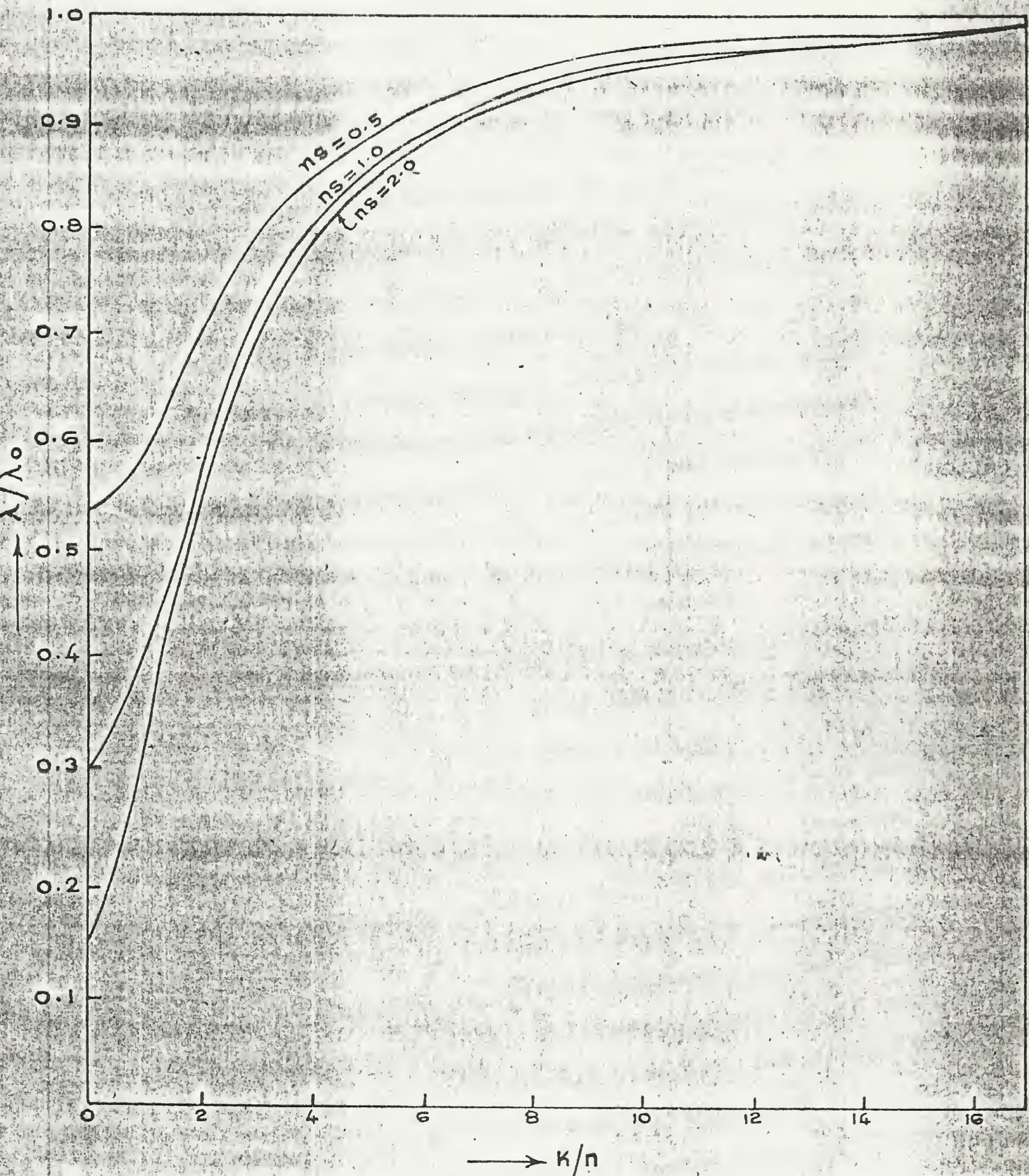

Fig. 4.4. Corrections in natural frequencies of a simply supported beam owing to shear deformation (d=0)

this graph, the value of the longitudinal inertia parameter d is taken as equal to zero.

For example, if we consider the variation of λ/λ_o for the fundamental mode of vibration (ie., n = 1), we can observe from Fig.4.4 that for a value of K = 1, and for s = 2.0, the value of λ/λ_o is 0.34 which means that the reduction in the value of the torsional frequency is by 66 percent. It can be therefore stated that for any constant values of n and K, the increase in the values of shear parameter s decreases the values of λ/λ_o (ie., the modifying quotient q). This reduction can be seen to be profound for smaller values of K and for higher modes of vibration (ie., for larger values of n). If the value of shear parameter s is taken as constant, say 0.5, it can be observed from Fig.4.4 that for K = 4.0 and n = 1, the value of λ/λ_o is 0.85 (ie., reduction is by 15 percent) and for K = 4.0, and n = 4, the value of λ/λ_o is 0.34 (ie., reduction is by 66 percent). It can be also observed that the increase in the value of mode number n and (or) decrease in the value of warping parameter K, decreases the values of λ/λ_o. It can be therefore concluded that the individual influence of shear deformation is to decrease the torsional frequency for any mode of vibration and that this reduction becomes significant for higher modes of vibration and for smaller values of warping parameter K (ie., for short beams). From Figs.4.3 and 4.4 we can observe that the effects of both longitudinal inertia and shear deformation is to decrease the frequency of vibration and that this reduction becomes significant for higher modes of vibration. It can be also observed that comparatively the individual influence of shear deformation on the torsional frequency of vibration is more profound than that of longitudinal inertia.

The combined effects of longitudinal inertia and shear deformation on the first four torsional frequencies of the first set of simply-supported, clamped-simply supported and clamped-clamped beams (s = 2d) are shown in Tables 4.1, 4.2 and 4.3 respectively. The values of the frequency parameter λ^2 and modified quotients $q = \lambda/\lambda_o$ for the first four modes of torsional vibration are given in these tables for various combinations of the parameters K, s and d.

It can be observed from Table 4.1 that in the case of simply-supported beams for K = 0.01, s = 0.10 and d = 0.05, the modifying quotients for the first four modes are respectively 0.944, 0.826, 0.705 and 0.603 and therefore the reductions in the first four torsional frequencies are respectively by 5.6%, 17.4%, 29.5% and 39.7%. For K = 10.0, s = 0.10 and d = 0.05, the modifying quotients for the first four modes are respectively 0.986, 0.934, 0.851 and 0.762 and therefore the reductions in the first four torsional frequencies are respectively by 1.4%, 6.6%, 14.9% and 23.8%. From these values we can observe that the increase in the value of warping parameter K reduces the effects of longitudinal inertia and shear deformation on the torsional frequencies of vibration and that for smaller values

TABLE - 4.1

Effects of longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams. (s = 2d)

K	s	d	Values of frequency parameter λ^2 and modifying quotients $q = \lambda / \lambda_o$: I Mode	q_1	II Mode	q_2	III Mode	q_3	IV MODE	q_4
0.01	0.00	0.00	97.411	1.000	1558.563	1.000	7890.216	1.000	24936.965	1.000
	0.04	0.02	95.559	0.990	1445.771	0.963	6724.678	0.923	19129.629	0.876
	0.08	0.04	90.361	0.963	1195.602	0.876	4747.525	0.776	11629.818	0.683
	0.10	0.05	86.882	0.944	1062.477	0.826	3920.978	0.705	9077.973	0.603
1.00	0.00	0.00	107.280	1.000	1598.038	1.000	7979.033	1.000	25094.863	1.000
	0.04	0.02	105.429	0.991	1484.710	0.964	6810.993	0.924	19281.160	0.877
	0.08	0.04	100.096	0.166	1233.507	0.879	4831.089	0.778	11777.295	0.685
	0.10	0.05	96.554	0.949	1099.974	0.830	4004.032	0.708	9225.332	0.606
10.00	0.00	0.00	1084.375	1.000	5506.419	1.000	16772.891	1.000	40728.391	1.000
	0.04	0.02	1078.618	0.997	5339.086	0.985	15368.900	0.957	34293.359	0.918
	0.08	0.04	1063.531	0.990	4982.955	0.951	13086.492	0.883	26296.227	0.804
	0.10	0.05	1053.563	0.986	4798.909	0.934	12158.678	0.851	23640.070	0.762

TABLE - 4.2

Effects of longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported thin-walled beams (s=2d).

K	s	d	I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
			Values of the frequency parameter λ^2 and modifying quotients $q = \lambda/\lambda_o$							
0.01	0.00	0.00	249.614	1.000	3993.813	1.000	20218.664	1.000	63900.938	1.000
	0.04	0.02	243.820	0.988	3642.962	0.955	16690.797	0.909	46820.211	0.856
	0.08	0.04	227.685	0.955	2926.263	0.856	11414.037	0.751	27857.102	0.660
	0.10	0.05	217.290	0.933	2572.443	0.803	9390.227	0.681	21881.023	0.585
1.00	0.00	0.00	261.950	1.000	4043.156	1.000	20329.684	1.000	64098.313	1.000
	0.04	0.02	256.114	0.989	3693.813	0.956	16809.074	0.909	47040.188	0.857
	0.08	0.04	240.398	0.958	2981.270	0.859	11551.523	0.754	28131.027	0.662
	0.10	0.05	230.203	0.937	2630.307	0.807	9539 .969	0.685	22188.258	0.588
10.00	0.00	0.00	1483.319	1.000	8928.631	1.000	31322.004	1.000	83640.219	1.000
	0.04	0.02	1486.950	1.001	8727.613	0.989	28538.156	0.955	68848.235	0.907
	0.08	0.04	1499.197	1.005	8452.291	0.973	25412.449	0.901	56551.453	0.822
	0.10	0.05	1510.454	1.009	8451.986	0.973	25375.574	0.900	59883.250	0.846

T A B L E - 4.3

Effects of Longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped thin-walled Beams (s=2d).

K	s	d	Values of λ^2_ϕ and λ/λ_0							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0.01	0.00	0.00	519.521	1.000	8312.322	1.000	42081.117	1.000	132997.094	1.000
	0.04	0.02	506.516	0.987	7553.774	0.953	34643.352	0.907	97904.031	0.858
	0.08	0.04	472.111	0.953	6119.002	0.858	24856.652	0.769	66324.172	0.706
	0.10	0.05	450.494	0.931	5463.667	0.811	21719.863	0.718	66035.985	0.705
1.00	0.00	0.00	532.679	1.000	8364.955	1.000	42199.539	1.000	133207.625	1.000
	0.04	0.02	520.175	0.988	7613.752	0.954	34796.836	0.908	98226.422	0.859
	0.08	0.04	487.097	0.956	6198.148	0.861	25105.473	0.771	67019.500	0.709
	0.10	0.05	466.436	0.936	5556.567	0.815	22061.781	0.723	63261.859	0.689
10.00	0.00	0.00	1835.473	1.000	13576.129	1.000	53924.686	1.000	154052.313	1.000
	0.04	0.02	1870.097	1.009	13551.494	0.999	52975.867	0.991	129726.219	0.918
	0.08	0.04	1973.504	1.037	14213.285	1.023	50029.805	0.963	84112 .531	0.739
	0.10	0.05	2054.938	1.058	15654.676	1.074	28024.945	0.721	15597.772	0.318

T A B L E - 4.4

Values of the Second set of first ~~five~~ four torsional frequencies of simply supported thin-walled beams (s=2d).

- K	s	d	Values of second set of λ^2.			
			I Mode	II Mode	III Mode	IV Mode
0.01	0.04	0.02	1593247.253	1684425.253	1833359.503	2036853.003
	0.08	0.04	105276.578	127303.313	162304.813	209397.000
	0.10	0.05	44847.953	58676.852	80492.469	109879.281
1.00	0.04	0.02	1593247.253	1684425.503	1833361.753	2036859.503
	0.08	0.04	105276.688	127304.875	162309 .969	209407.438
	0.10	0.05	44848.156	58678.828	80498.235	109889.797
10.00	0.04	0.02	1593251.003	1684479.753	1833597.753	2037480.503
	0.08	0.04	105290.313	127463.813	162848.407	210522.000
	0.10	0.05	44868.242	58888.274	81137.438	111108.594

TABLE - 4.5

Values of the Second set of first ~~five~~ four torsional frequencies of clamped-simply supported thin-walled beams (s=2d).

K	s	d	Values of Second set of λ^2			
			I Mode	II Mode	III Mode	IV Mode
0.01	0.04	0.02	1600809.503	1713070.503	1892789.253	2132533.007
	0.08	0.04	107066.906	133283.313	172987.188	224012.407
	0.10	0.05	45951.258	62101.703	86126.594	116815.516
1.00	0.04	0.02	1600809.503	1713069.003	1892782.003	2132510.506
	0.08	0.04	107066.531	133277.657	172960.719	223935.844
	0.10	0.05	45950.680	62093.180	86087.875	116705.656
10.00	0.04	0.02	1600800.253	1712920.753	1892045.003	2130244.506
	0.08	0.04	107029.125	132692.125	170092.125	215057.313
	0.10	0.05	45891.797	61156.977	81244.578	98552.562

TABLE - 4.6

Values of the Second set of first four torsional frequencies of clamped-clamped thin-walled beams (s=2d).

K	s	d	Values of Second set of λ^2			
			I Mode	II Mode	III Mode	IV Mode
0.01	0.04	0.02	1603117.503	1719440.753	1897969.003	2122573.006
	0.08	0.04	107465.047	132660.813	165327.563	195826.219
	0.10	0.05	46129.297	60855.430	77498.063	79240.328
1.00	0.04	0.02	1603117.003	1719433.503	1897934.003	2122467.507
	0.08	0.04	107463.235	132634.282	165197.188	195341.469
	0.10	0.05	46126.516	60815.164	77274.578	82224.969
10.00	0.04	0.02	1603070.003	1718707.003	1894426.003	2111805.507
	0.08	0.04	107279.625	129830.344	149051.907	199093.094
	0.10	0.05	45840.797	55928.227	83036.547	150733.750

of K the reductions in the torsional frequencies at higher modes owing to these second order effects become quite significant and should be taken care of. Similar observations can be made from Tables 4.2 and 4.3 for clamped-simply supported and clamped-clamped beams. It can be also noticed that these reductions in the torsional frequencies due to longitudinal inertia and shear deformation are comparatively high in the case of clamped-clamped beams than in the case of clamped-simply supported or simply-supported beams.

The results for the second set of frequencies for the simply supported, clamped-simply supported and clamped-clamped beams are given in Tables 4.4, 4.5 and 4.6 respectively. It must be recalled here that these second set of frequencies exist solely due to the inclusion of these second order effects. From Tables 4.4 to 4.6, we observe that even in the case of second set, the effect of increase in the values of the parameters s and d is to reduce significantly the frequencies at higher modes of vibration. It is interesting to note that the increase in the value of the warping parameter K is having a negligible effect on these reductions in the frequencies of the second set for all the three boundary conditions considered here.

CHAPTER - V

FINITE ELEMENT ANALYSIS OF TORSIONAL VIBRATIONS OF SHORT WIDE-FLANGED THIN-WALLED BEAMS INCLUDING THE EFFECTS OF LONGITUDINAL INERTIA AND SHEAR DEFORMATION*.

5.1. INTRODUCTION:

The problem of torsional vibrations of short wide-flanged thin-walled beams including the effects of longitudinal inertia and shear deformation is completely solved in Chapter IV utilizing rigorous mathematical analysis. The highly transcendental frequency equations obtained for various end conditions could be solved only by lengthy trial-and-error procedure. Except for the case of simply-supported beam, the results for other complex boundary conditions could be obtained only by expending considerable effort.

Even the approximate analytical methods such as Ritz and Galerkin techniques have a tendency to become very tedious for some complex boundary conditions. The complexity of the analytical techniques even for simple end conditions emphasizes the need for physically satisfactory approximate solutions. To this end, the present Chapter aims at developing a finite element analysis of torsional vibrations of short wide-flanged thin-walled beams including the effects of longitudinal inertia and shear deformation.

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* A paper by the author based on the results from this Chapter is accepted for publication in AIAA Journal, See Ref.(52).

The basic theory behind the finite element method for dynamic problems is briefly presented in Chapter III and is shown to give results which are in excellent agreement with the exact ones. This chapter, therefore, extends the finite element method to torsional vibrations of doubly-symmetric thin-walled beams of open section including the effects of longitudinal inertia and shear deformation. New stiffness and mass matrices for a thin-walled beam are developed in this chapter, for the first time and, to the best of author's knowledge, there is no other finite element formulation for this problem available in the literature. The method developed in this chapter is applicable to uniform as well as non-uniform beams with any complex boundary conditions. A consistant mass matrix is made use of in conjunction with the corresponding stiffness matrix for finding the frequencies and mode shapes for free torsional vibrations of uniform thin-walled beams with various boundary conditions. Results obtained are compared with the exact ones obtained in Chapter IV and an excellent agreement is observed.

5.2. MODIFIED ENERGY EXPRESSIONS:

Two approaches are made to our present problem. In the first approach, the stiffness and mass matrices are developed in terms of the total angle of twist ϕ and the warping angle directly utilizing the strain and kinetic energy expressions (Eqs.4.12 and 4.13) derived in Chapter IV. By assuming only one degree of freedom for each of the angles ϕ and ψ, the stiffness and mass matrices each of 4 x 4 size are obtained which include the second order effects. But the matrices obtained in this approach, though not shown here, does not satisfy the exact boundary conditions and thus could not yield good results.

An alternative approach which will be discussed in detail in this chapter is to split the total angle of twist into two parts: One part is the twist calculated by neglecting the shear strain in the strain energy expression, (Eq.(4.12)); and the second part gives the contribution due to shear strain.

Let us define the total angle of twist ϕ as:

$$\phi(z,t) = \phi_t(z,t) + \phi_s(z,t) \qquad (5.1)$$

where the subscript "t" denotes the part of the solution when the shear strain has been neglected, and the subscript s denotes the contribution of the shear strain to the total angle of twist. This type of choice has the advantage that when ϕ_s is equated to zero, the resulting expressions reduce back to the equations for the lengthy beams presented and solved in Chapter-II. This approach is quite convenient as it satisfactorily encompasses all boundary conditions of the present problem.

By substituting Eq.(5.1) into Eq.(4.9) we obtain:

$$u = (h/2)\ (\phi_t + \phi_s) \qquad (5.2)$$

Substituting of Eq.(5.2) into Eq.(4.6) gives:

$$\psi + \epsilon_{sh} = \frac{h}{2}\frac{\partial \phi_t}{\partial z} + \frac{h}{2}\frac{\partial \phi_s}{\partial z} \qquad (5.3)$$

From Eq.(5.3) we can write:

$$\psi = \frac{h}{2}\frac{\partial \phi_t}{\partial z} \qquad (5.4)$$

and

$$\varepsilon_{sh} = \frac{h}{2} \frac{\partial \phi_s}{\partial z} \qquad (5.5)$$

By substituting the expressions for ψ and ε_{sh} from Eqs.(5.4) and (5.5) respectively into Eqs.(4.4) and (4.7), the expressions for moment M and shear force Q can be obtained as:

$$M = EI_f \frac{h}{2} \frac{\partial^2 \phi_t}{\partial z^2} \qquad (5.6)$$

and

$$- Q = K' A_f G \frac{h}{2} \frac{\partial \phi_s}{\partial z} \qquad (5.7)$$

By substituting Eq.(5.1) into Eq.(4.1), the strain energy U_1 due to saint-venant torsion can be obtained as:

$$U_1 = \frac{1}{2} \int_0^L GC_s \left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z}\right)^2 dz \qquad (5.8)$$

By substituting Eqs.(5.6) and (5.4) into Eq.(4.5), the strain energy U_2 of the two flanges due to warping normal strain becomes:

$$U_2 = \frac{1}{2} \int_0^L EC_w \left(\frac{\partial^2 \phi_t}{\partial z^2}\right)^2 dz \qquad (5.9)$$

Substituting Eqs.(5.1) and (5.7) into Eqs.(2.2a) and (4.8), the expressions for the Saint-Venant torque T_s and the torque due to warping T_w can be respectively obtained as:

$$T_s = GC_s \left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z} \right) \qquad (5.10)$$

and

$$T_w = - Qh = K' A_f G \frac{h^2}{2} \frac{\partial \phi_s}{\partial z} \qquad (5.11)$$

Hence the total torque T_t (See Eq.4.10) can be obtained from Eqs.(5.10) and (5.11) as:

$$T_t = GC_s\left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z}\right) + K' A_f G \frac{h^2}{2} \frac{\partial \phi_s}{\partial z} \tag{5.12}$$

Substituting Eqs.(5.7) and (5.5) into Eq.(4.11), the strain energy due to shear deformation of the two flanges, U_3, becomes:

$$U_3 = \frac{1}{2} \int_0^L K' A_f G \frac{h^2}{2} \left(\frac{\partial \phi_s}{\partial z}\right)^2 dz \tag{5.13}$$

The total strain energy, U, at any instant t (See Eq. 4.12) is the sum of the energies U_1, U_2 and U_3 and therefore given by

$$U = \frac{1}{2} \int_0^L \left[GC_s\left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z}\right)^2 + EC_w\left(\frac{\partial^2 \phi_t}{\partial z^2}\right)^2 + K' A_f G \frac{h^2}{2} \left(\frac{\partial \phi_s}{\partial z}\right)^2 \right] dz \tag{5.14}$$

By substituting Eqs.(5.1) and (5.4) into Eq.(4.13), the total kinetic energy, T, at time t becomes:

$$T = \frac{1}{2} \int_0^L \left[\rho I_p\left(\frac{\partial \phi_t}{\partial t} + \frac{\partial \phi_s}{\partial t}\right)^2 + \rho C_w\left(\frac{\partial^2 \phi_t}{\partial z \partial t}\right)^2 \right] dz \tag{5.15}$$

5.3. MODIFIED NATURAL BOUNDARY CONDITIONS:

In terms of the angles ϕ_t and ϕ_s the natural boundary conditions given by Eqs.(4.19) to (4.22) can be modified as follows:

(a) Simply supported end:

$$\phi_s = 0; \quad \phi_t = 0; \quad \frac{\partial^2 \phi_t}{\partial z^2} = 0 \tag{5.16}$$

(b) Fixed end:

$$\phi_s = 0; \quad \phi_t = 0; \quad \frac{\partial \phi_t}{\partial z} = 0 \qquad (5.17)$$

(c) Free end:

$$\frac{\partial^2 \phi_t}{\partial z^2} = 0; \quad GC_s \frac{\partial \phi_t}{\partial z} + (GC_s + K' A_f G\, h^2/2) \frac{\partial \phi_s}{\partial z} = 0 \qquad (5.18)$$

or

$$\frac{\partial \phi_t}{\partial z} = 0 \; ; \quad \frac{\partial \phi_s}{\partial z} = 0 \qquad (5.19)$$

The conditions given by Eq. (5.19) are useful for finding symmetric modes of vibration in simply supported, fixed-fixed and free-free beams.

5.4. FINITE ELEMENT FORMULATION:

In the present formulation, for each finite element of a short thin-walled beam in torsion including the effects of longitudinal inertia and shear deformation in addition to warping, there are four generalized nodal displacements at the j end of the ith member. These nodal displacements are:

ϕ_{tj} = angle of twist neglecting shear strain at the shear center about z-axis;

ϕ'_{tj} = rate of change of ϕ_t at the shear center about z-axis;

ϕ_{sj} = angle of twist due to shear strain at the shear center about z-axis;

ϕ'_{sj} = rate of change of ϕ_s at the shear center about z-axis;

where subscript j denotes the generalized displacement at the j end of the ith finite element. Similar generalized nodal displacements exist at the K end of the element. The prime denotes differentiation with respect to z.

Assuming the angles ϕ_t and ϕ_s within each finite element to vary cubically the displacement functions take the form:

$$\phi_t(z) = a_1 + b_1 z + c_1 z^2 + d_1 z^3 \qquad (5.20)$$

and

$$\phi_s(z) = a_2 + b_2 z + c_2 z^2 + d_2 z^3 \qquad (5.21)$$

To establish relationships between the displacements at any interior coordinate z in terms of the generalized nodal coordinates, the eight arbitrary constants in the assumed displacement functions must be determined.

After determining the coefficients in Eqs.(5.20) and (5.21), the angles ϕ_t and ϕ_s at any coordinate z within the element in terms of the nodal displacements ϕ_{tj}, $\partial\phi_{tj}/\partial z$, ϕ_{tK}, and $\partial\phi_{tK}/\partial z$ and, ϕ_{sj}, $\partial\phi_{sj}/\partial z$, ϕ_{sK}, and $\partial\phi_{sK}/\partial z$ can be respectively defined as follows:

$$\phi_t(z) = \left[(1-3\bar{\xi}_1^2 + 2\bar{\xi}_1^3),\ z(1-2\bar{\xi}_1+\bar{\xi}_1^2),\ (3\bar{\xi}_1^2-2\bar{\xi}_1^3),\ z(-\bar{\xi}_1+\bar{\xi}_1^2)\right]\bar{R}_{tN}(t) \qquad (5.22)$$

and

$$\phi_s(z)=\left[(1-3\bar{\xi}_1^2+2\bar{\xi}_1^3),z(1-2\bar{\xi}_1+\bar{\xi}_1^2),(3\bar{\xi}_1^2-2\bar{\xi}_1^3),z(-\bar{\xi}_1+\bar{\xi}_1^2)\right]\bar{R}_{sN}(t) \tag{5.23}$$

where $\bar{\xi}_1 = z/l$.

Eqs.(5.22) and (5.23) can be written in an abbreviated form as follows:

$$\phi_t(z) = \bar{A}(z)\,\bar{R}_{tN}(t) \tag{5.24}$$

and

$$\phi_s(z) = \bar{A}(z)\,\bar{R}_{sN}(t) \tag{5.25}$$

where

$$\bar{R}_{tN} = [\,\phi_{tj},\ \phi'_{tj},\ \phi_{tK},\ \phi'_{tK}\,] \tag{5.26}$$

$$\bar{R}_{sN} = [\,\phi_{sj},\ \phi'_{sj},\ \phi_{sK},\ \phi'_{sK}\,] \tag{5.27}$$

and $\bar{A}(z)$ is given by Eq.(3.23).

Similarly, for the first and second derivatives of the angles ϕ_t and ϕ_s, the matrix relations can be written as:

$$\phi'_t(z) = (\bar{A}(z)\bar{R}_{tN}(t))' = \bar{A}_1(z)\bar{R}_{tN}(t) \tag{5.28}$$

$$\phi''_t(z) = (\bar{A}(z)\bar{R}_{tN}(t))'' = \bar{A}_2(z)\bar{R}_{tN}(t) \tag{5.29}$$

$$\phi'_s(z) = (\bar{A}(z)\bar{R}_{sN}(t))' = \bar{A}_1(z)\bar{R}_{sN}(t) \tag{5.30}$$

and

$$\phi''_s(z) = (\bar{A}(z)\bar{R}_{sN}(t))'' = \bar{A}_2(z)\bar{R}_{sN}(t) \tag{5.31}$$

where $\bar{A}_1(z)$ and $\bar{A}_2(z)$ are defined by Eqs.(3.27) and (3.28).

The generalized velocities and accelerations can also be expressed in terms of the discretized nodal velocities and accelerations:

That is:

$$\dot{\phi}_t(z) = \bar{A}(z)\,\dot{\bar{R}}_{tN}(t) \qquad (5.32)$$

$$\dot{\phi}'_t(z) = \bar{A}_1(z)\,\dot{\bar{R}}_{tN}(t) \qquad (5.33)$$

$$\ddot{\phi}_t(z) = \bar{A}(z)\,\ddot{\bar{R}}_{tN}(t) \qquad (5.34)$$

$$\dot{\phi}_s(z) = \bar{A}(z)\,\dot{\bar{R}}_{sN}(t) \qquad (5.35)$$

and

$$\ddot{\phi}_s(z) = \bar{A}(z)\,\ddot{\bar{R}}_{sN}(t) \qquad (5.36)$$

where dots denote differentiation with respect to time t.

5.5. Derivation of Element Matrices including Second Order Effects:

The expressions for the strain energy U, and Kinetic energy T_K, given by Eqs.(5.14) and (5.15) respectively, for an element of finite length, l, can be written as follows:

$$U = \frac{1}{2}\int_0^l \left[GC_s(\phi'_t + \phi'_s)^2 + EC_w(\phi''_t)^2 + K'A_f G\frac{h^2}{2}(\phi'_s)^2 \right] dz \qquad (5.37)$$

and

$$T = \frac{1}{2}\int_0^l \left[\rho I_p(\dot{\phi}_t + \dot{\phi}_s)^2 + \rho C_w(\dot{\phi}'_t)^2 \right] dz \qquad (5.38)$$

Direct substitution of Eqs.(5.24) to (5.36) into Eqs.(5.37) and (5.38) and the resulting expressions into ^(FE eqns. representing) Hamilton's Principle, Eq.(3.34) for $W = 0$, yields (for the Nth element):

$$\delta I_N = \delta \int_{t_1}^{t_2} \left\{ \frac{\rho I_p}{2} \left[\int_0^1 \dot{\bar{R}}_{tN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{tN}\, dz + \int_0^1 \dot{\bar{R}}_{sN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{sN}\, dz \right.\right.$$

$$\left. + \int_0^1 \dot{\bar{R}}_{tN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{sN}\, dz + \int_0^1 \dot{\bar{R}}_{sN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{tN}\, dz \right]$$

$$+ \frac{\rho C_w}{2} \int_0^1 \dot{\bar{R}}_{tN}^T \bar{A}_1^T \bar{A} \dot{\bar{R}}_{tN}\, dz$$

$$- \frac{1}{2} \int_0^1 \bar{R}_{tN}^T \left[EC_w \bar{A}_2^T \bar{A}_2 + GC_s \bar{A}_1^T \bar{A}_1 \right] \bar{R}_{tN}\, dz$$

$$- \frac{1}{2} \left(GC_s + K' A_f G \frac{h^2}{2} \right) \int_0^1 \bar{R}_{sN}^T \bar{A}_1^T \bar{A}_1 \bar{R}_{sN}\, dz$$

$$\left. - \frac{GC_s}{2} \left[\int_0^1 \bar{R}_{tN}^T \bar{A}_1^T \bar{A}_1 \bar{R}_{sN}\, dz + \int_0^1 \bar{R}_{sN}^T \bar{A}_1^T \bar{A}_1 \bar{R}_{tN}\, dz \right] \right\} dt$$

$$= 0 \qquad (5.39)$$

Eq.(5.39) can be also written more concisely as follows:

$$\delta I_N = \delta \int_{t_1}^{t_2} \frac{1}{2} \left[(\rho I_p L)\, \dot{\bar{q}}_N^T m_N \dot{\bar{q}}_N - (EC_w/L^3) \bar{q}_N^T \bar{k}_N \bar{q}_N \right] dt = 0 \qquad (5.40)$$

In Eq.(5.40) the terms $(\rho I_p L) m_N$ and $(EC_w/L^3)\bar{k}_N$ denote respectively the new mass and stiffness matrices $\bar{M}_N$ and $\bar{K}_N$ respectively of the Nth element. The matrices $\bar{m}_N$, $\bar{K}_N$ and $\bar{q}_N$ are given below:

$$m_N = \frac{1}{420\, N^4} \begin{bmatrix} \bar{m}_{11} & \bar{m}_{21}^T \\ \bar{m}_{21} & \bar{m}_{22} \end{bmatrix} \qquad (5.41)$$

$$\bar{K}_N = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{21}^T \\ \bar{K}_{21} & \bar{K}_{22} \end{bmatrix} \tag{5.42}$$

and

$$\bar{q}_N = \left[\bar{q}_{tN}, \quad \bar{q}_{sN} \right] \tag{5.43}$$

where

$$\bar{m}_{11} = \frac{1}{420N^4} \begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} + \frac{d^2N^2}{30} \begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \tag{5.44}$$

$$\bar{m}_{21} = \bar{m}_{22} = \frac{1}{420N^4} \begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \tag{5.45}$$

$$\bar{K}_{11} = \begin{bmatrix} 12N^2 & & \text{Sym.} & \\ 6N & 4 & & \\ -12N^2 & -6N & 12N^2 & \\ 6N & 2 & -6N & 4 \end{bmatrix}$$

$$+\ \frac{K^2}{30N^2}\begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \qquad (5.46)$$

$$\bar{K}_{21} = \frac{K^2}{30N^2}\begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \qquad (5.47)$$

$$\bar{K}_{22} = \frac{(s^2K^2+1)}{30\ s^2N^2}\begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \qquad (5.48)$$

$$\bar{q}_{tN} = [\, \phi_{tj},\ L\phi'_{tj},\ \phi_{tK},\ L\phi'_{tK} \,] \qquad (5.49)$$

$$\bar{q}_{sN} = [\, \phi_{sj},\ L\phi'_{sj},\ \phi_{sK},\ L\phi'_{sK} \,] \qquad (5.50)$$

and the non-dimensional parameters K^2, d^2 and s^2 are previously defined by Eqs.(4.39), (4.40), and (4.41) respectively.

The equations of motion for the discretized system can now be obtained using Eq.(5.40). Taking the variation of the integral expression of Eq.(5.40) we obtain:

$$\int_{t_1}^{t_2} [\ (\rho I_p L)\ \bar{\delta}\ \dot{\bar{q}}_N^T\ \bar{m}_N\ \dot{\bar{q}}_N - (EC_w/L^3)\ \bar{\delta}\ \bar{q}_N^T\ \bar{K}_n\ \bar{q}_N\ ]\ dt = 0 \qquad (5.51)$$

which after integration by parts over the time interval gives:

$$(\rho I_p L)\, \bar{\delta}\, \bar{\dot{q}}_N^T\, \bar{m}_N\, \bar{\dot{q}}_N \Big|_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} \bar{\delta}\, \bar{q}_N^T \left[(\rho I_p L)\, \bar{m}_N\, \ddot{\bar{q}}_N + (EC_w/L^3)\, \bar{K}_N\, \bar{q}_N \right] dt = 0 \qquad (5.52)$$

The first term in Eq.(5.52) is seen to vanish in view of the assumptions made previously that the virtual displacements $\bar{\delta}\bar{q}_N$ are zero at the time instants t_1 and t_2. Since the virtual displacements can be arbitrary for other times then the only way in which the integral expression in Eq.(5.52) can vanish is for the terms within the brackets to equal zero. Therefore, the governing dynamic equilibrium equations for the discretized systems are:

$$(\rho I_p L)\bar{m}_N \ddot{\bar{q}}_N + (EC_w/L^3)\, \bar{K}_N \bar{q}_N = 0 \qquad (5.53)$$

Assuming that the displacements undergo harmonic oscillation, the displacement vector $\bar{q}_N$ can be written as:

$$q_N = \bar{Q}_N\, e^{ip_n t} \qquad (5.54)$$

where $\bar{Q}_N$ is a column vector of torsional amplitudes of the general torsional displacements. Substituting Eq.(5.54) into (5.53) gives:

$$\left[(EC_w/L^3)\bar{K}_N - (\rho I_p L\, p_n^2)\bar{m}_N \right] \bar{Q}_N\, e^{ip_n t} = 0 \qquad (5.55)$$

Deviding throughout by EC_w/L^3 and cancelling $e^{ip_n t}$, Eq.(5.55) becomes

$$[\bar{K}_N][\bar{Q}_N] = \lambda^2 [\bar{m}_N][\bar{Q}_N] \qquad (5.56)$$

where λ^2 is the non-dimensional frequency parameter defined previously by (Eq.(4.38). Eq.(5.56) represents the equations of motion for an undamped free oscillating system including the effects of longitudinal inertia and shear deformation.

5.6. Equations of Equilibrium for the totally assembled beam:

Following the procedure outlined in section 3.5 and utilising the element stiffness and mass matrices presented in section 5.5, the equations of equilibrium for the totally assembled beam can be obtained as:

$$[\bar{K}][\bar{Q}] = \lambda^2 [\bar{m}][\bar{Q}] \qquad (5.57)$$

where $\bar{K}$, $\bar{m}$ and $\bar{Q}$ denote the totally assembled matrices corresponding to the element matrices $\bar{K}_N$, $\bar{m}_N$ and $\bar{Q}_N$ defined previously. With the four generalized displacements possible at each node and with the bar segmented into N elements, the total number of degrees of freedom is 4 (N+1). The formulation of the matrix equilibrium equation, Eq.(5.57), includes all possible degrees of freedom, both free and restrained. The displacement vector Q of this overall joint equilibrium equations is comprized of both degrees of freedom, the unknowns of the problem and known support displacements or boundary conditions.

5.7. Boundary conditions useful for Modifying the total Matrices:

It should be recalled here that for the present finite element formulation, totally four generalized displacements are considered at each node. The following are therefore the boundary conditions to be utilized in order to modify the total stiffness and mass matrices for various combinations of end supports.

(a) Simply supported end:

$$\phi_s = 0 \; ; \; \phi_t = 0 \qquad (5.58)$$

(b) Fixed end:

$$\phi_s = 0 \; ; \; \phi_t = 0 \; ; \; L\phi_t' = 0 \qquad (5.59)$$

(c) Free end:

The total matrices need not be modified in this case.

(d) $$L\,\phi_t' = 0 \; ; \; L\,\phi_s' = 0 \qquad (5.60)$$

Eqs. (5.58) to (5.60) are useful for finding symmetric modes of vibration in simply supported, fixed-fixed and free-free beams.

5.8. RESULTS AND CONCLUSIONS:

A digital computer programme is written in Fortran IV which can give results for any set of boundary conditions. Results for simply supported and fixed-fixed beams for values of K = 1.541, s = 0.046 and d = 0.023, are obtained on IBM 1130 Computer at Andhra University, Waltair and are presented in Tables 5.1 and 5.2.

For the simply supported case, the first and second sets of values of λ obtained for the first four modes of vibration for a division of the beam into N = 2 and 3 segments are shown in Table 5.1 and are compared with the exact results obtained using the analysis presented in Chapter IV. For, the fixed-fixed beam, the first set of values of λ obtained for the first four modes of vibration of N = 2 and 3 are shown in Table 5.2 and are compared with the exact results. The exact results for the simply supported case were obtained using Eq.(4.65) and for the fixed-fixed beam, the results were obtained using Eqs.(4.44) and (4.72).

It can be seen from Tables 5.1 and 5.2 that for all cases, excellent results have been obtained even for very coarse subdivisions of the beam. Since the stiffness and mass matrices including shear deformation and longitudinal inertia seperately involve double the number of degrees of freedom than those that exist if they are neglected, twice as many natural frequencies result. In Table 5.1 the lower and higher spectrum of frequen-

T A B L E - 5.1

Comparison of first and second sets of values of λ from the Finite element Method and those from exact analysis given in Chapter IV for a simply supported beam (K=1.541, s=0.046, d=0.023).

Mode	Exact Values of λ from Chap.IV	No.of elements and % error					
		One element	% error	Two elements	% error	Three elements	% error
First Set:							
I	10.8722	11.7421	8.01%	11.1132	2.2%	10.8814	0.08%
II	38.7942	47.9234	23.54%	42.2221	8.84%	38.9231	0.33%
III	81.3913			108.1012	32.82%	96.9422	19.10%
IV	134.8025			161.4034	19.73%	151.3014	12.24%
V	195.6023					240.7015	23.06%
Second Set:							
I	962.54	964.72	0.23%	963.44	0.09%	962.73	0.02%
II	998.22	1018.43	2.03%	1007.23	0.90%	999.35	0.11%
III	1053.37			1093.14	3.78%	1072.06	1.78%
IV	1124.52			1191.38	5.93%	1165.17	3.60%
V	1207.32					1317.43	

T A B L E - 5.2

Comparison of the first set of values of λ from the finite element method and those from exact analysis given in Chapter IV for a fixed-fixed beam (K=1.541, s= 0.046, d=0.023).

Mode	Exact Values of λ from Chap.IV	No.of elements and % error			
		Two elements	% error	Three elements	% error
I	21.6699	21.8663	0.91%	21.8374	0.78%
II	55.9769	69.3964	23.94%	67.8850	21.24%
III	101.7908	185.9526	82.96%	116.5183	14.47%
IV	155.7791	241.3891	54.96%	194.7396	25.01%
V	215.4931			303.6783	40.93%

cies obtained can also be observed to be in excellent agreement with the exact ones. In Chapter IV, we have discussed this second set of frequencies in detail.

Using the above stiffness and mass matrices, beams with various other boundary conditions, can be analyzed easily. A beam with variable cross section can also be analyzed by dividing the beam into a number of segments and assuming that each segment has a constant cross section. In all cases (as we observed from Tables 5.1 and 5.2), the method gives an upper bound to the exact frequencies of the system. The approach presented in the Chapter is quite general, satisfactorily encompasses all boundary conditions and can be extended to static and dynamic stability of uniform and tapered thin-walled beams.

CHAPTER - VI

FORCED TORSIONAL VIBRATIONS OF SHORT WIDE-FLANGED BEAMS WITH LONGITUDINAL INERTIA, SHEAR DEFORMATION AND VISCOUS DAMPING*.

6.1. INTRODUCTION:

In Chapters IV and V, the problem of free torsional vibrations of short thin-walled beams of open section, including the effects of longitudinal inertia and shear deformation is completely analyzed utilizing the exact and approximate analytical methods and the powerful finite-element technique.

With regards to the forced torsional vibrations of thin-walled beams of open section very few studies are available in the literature. Tso (104), extended the Timoshenko torsion theory for coupled flexural-torsional vibrations of thin-walled beams of open sections and presented a formal solution to Gere's theory (32) under general loading conditions and general boundary conditions. Aggarwal (3), considered the problem of forced torsional vibrations of thin-walled beams of open section under very general loads including the effects of longitudinal inertia and shear deformation, and solved the specific case of a simply supported beam with a step torque impulsively applied at the mid-point. He compared the results obtained for the above problem, with those obtained utilizing Timoshenko torsion theory. But in all these studies the effect of damping is not

* A paper by the author, abstracted from this Chapter, is accepted for publication in the August 1976 issue of the Journal of the Aeronautical Society of India. See Ref. (53)

considered.

The present Chapter therefore deals with the study of forced torsional vibrations of doubly-symmetric thin-walled beams of open section such as an I-beam, including the effects of longitudinal inertia, shear deformation and viscous damping. Viscous damping forces arising separately from torsional and warping velocities are included in the equations of motion and the coupled fundamental equations of motion are formulated in terms of angle of twist and warping angle. The method of solution is demonstrated for arbitrary external torque for the beam having both ends simply-supported and numerical results are presented for the case when the torque is uniform over the span and varies sinusoidally in time. Amplitude response is plotted versus torsional frequency for varying amounts of torsional and warping damping, and is compared to the response for the classical beam (based on Timoshenko torsion theory) for the first five symmetric mode shapes.

6.2. DERIVATION OF EQUATIONS OF MOTION INCLUDING VISCOUS DAMPING:

In Fig.6.1, a typical differential element of length dz and width b_f is taken from the flange of the thin-walled beam, and the generalized forces acting are shown. Assuming small displacements as in Chapter IV and summing the torques yields one equation of motion:

$$\frac{\partial}{\partial z}(T_s + T_w) - \beta_t \frac{\partial \phi}{\partial t} + T_e = \rho I_p \frac{\partial^2 \phi}{\partial t^2} \qquad (6.1)$$

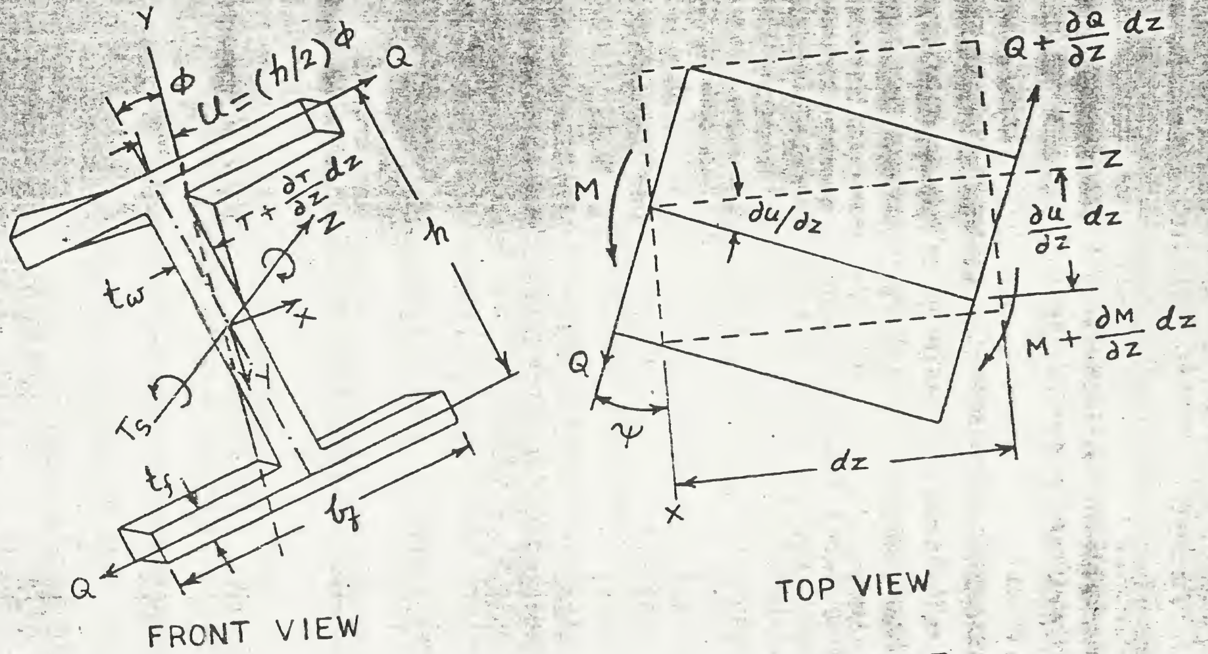

FIG. 6.1. STRAINED STATE OF A BEAM ELEMENT

where T_s is the Saint Venant torque given by Eq.(2.2a), T_w the warping torque given by Eq.(4.8), β_t the torsional damping constant and, T_e the external torque per unit length of the beam.

Summing moments about an axis normal to Fig.6.1 yields the second equation of motion:

$$\frac{\partial M}{\partial z} - Q - qb_{\gamma} = \rho I_f \frac{\partial^2 \psi}{\partial t^2} \tag{6.2}$$

where M is the bending moment in the top flange given by Eq.(4.4), Q the shear force given by Eq.(4.7), q the external viscous force per unit length acting along the sides of the flanges, of width b, to oppose warping.

Further, let us define a warping damping constant β_w by:

$$q = \frac{\beta_w}{b_{\gamma}} \frac{\partial \psi}{\partial t} \tag{6.3}$$

Substituting Eqs.(2.2a), (4.8), (4.4), (4.7) and (6.3) in Eqs.(6.1) and (6.2) we obtain:

$$GC_s \frac{\partial^2 \phi}{\partial z^2} + K' A_f Gh\left(\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z}\right) + T_e = \rho I_p \frac{\partial^2 \phi}{\partial t^2} + \beta_t \frac{\partial \phi}{\partial t} \tag{6.4}$$

and

$$EI_f \frac{\partial^2 \psi}{\partial t^2} + K' A_f G\left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi\right) = \rho I_f \frac{\partial^2 \psi}{\partial t^2} + \beta_w \frac{\partial \psi}{\partial t} \tag{6.5}$$

It is necessary to obtain solutions to the differential Equations (6.4) and (6.5) which also satisfy the boundary conditions of the particular problem being considered. This may be achieved by assuming solutions in the form:

$$\phi(z,\ t) = \sum_n \bar{\phi}_n(z)\, F_n(t) \tag{6.6}$$

$$\psi(z,\ t) = \sum_n \bar{\psi}_n(z)\, G_n(t) \tag{6.7}$$

where $\bar{\phi}_n(z)$ and $\bar{\psi}_n(z)$ are the mode shapes obtained from solving the free, undamped vibration problem. The mode shape functions are given in section 4.7 of Chapter IV for the six cases arising from combinations of simply supported, clamped and free ends. This procedure will be used below to investigate the case when both ends are simply supported.

6.3. SOLUTION FOR THE CASE OF A SIMPLY SUPPORTED BEAM:

Consider a beam of length L having its ends z=0 and z=L both simply supported. From Eq.(4.65) of Chapter IV, the frequencies of vibration for this case are given in an alternative form as:

$$p_n^2 = \frac{-\bar{b} \pm (\bar{b}^2 - 4\bar{a}\bar{c})^{1/2}}{2a} \tag{6.8}$$

where

$$a = \frac{\rho I_p \rho I_f L^4}{K' A_f G} \tag{6.9}$$

$$\bar{b} = -\left[\rho I_p L^4 + n^2\pi^2 L^2 \left(\frac{\rho I_p I_f}{K' A_f G}\right) + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right)\right] \tag{6.10}$$

$$\bar{c} = n^2\pi^2 L^2 G C_s + n^4\pi^4 \left(\frac{E I_f C_s}{K' A_f} + E C_w\right) \tag{6.11}$$

From Eqs.(4.67) and (4.68) of Chapter IV, the mode shapes for this case are given by:

$$\bar{\phi}_n(z) = A_n \sin \frac{n\pi z}{L} \tag{6.12}$$

$$\bar{\psi}_n(z) = B_n \cos \frac{n\pi z}{L} \tag{6.13}$$

where A_n and B_n are arbitrary amplitudes.

Let the external torque per unit length be expressed as:

$$T_e(z, t) = \sum_{n=1}^{\infty} \tau_n(t) \sin \frac{n\pi z}{L} \tag{6.14}$$

where Fourier coefficients are determined from

$$\tau_n(t) = \frac{2}{L} \int_0^L T_e(z,t) \sin \frac{n\pi z}{L}\, dz \tag{6.15}$$

The solution of the coupled differential Eqs.(6.4) and (6.5) can progress in several ways. We will begin by first uncoupling them. Differentiating Eq.(6.4) with respect to z, solving Eq.(6.4) for $\partial\psi/\partial z$, and its higher derivatives, and substituting into Eq.(6.5) yields a fourth order uncoupled equation for ϕ given by:

$$\left[\frac{EI_f C_s}{K' A_f} + EC_w\right] \frac{\partial^4 \phi}{\partial z^4} - \left[\frac{E \rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right] \frac{\partial^4 \phi}{\partial z^2 \partial t^2}$$

$$- GC_s \frac{\partial^2 \phi}{\partial z^2} - \left[\frac{EI_f \beta_t}{K' A_f G} + \frac{\beta_w C_s}{K' A_f} + \frac{\beta_w h^2}{2}\right] \frac{\partial^3 \phi}{\partial z^2 \partial t}$$

$$+ \frac{\rho^2 I_p I_f}{K' A_f G} \frac{\partial^4 \phi}{\partial t^4} + \left[\frac{\rho I_f \beta_t}{K' A_f G} + \frac{\rho I_p \beta_w}{K' A_f G}\right] \frac{\partial^3 \phi}{\partial t^3} + \left[\rho I_p + \frac{\beta_t \beta_w}{K' A_f G}\right] \frac{\partial^2 \phi}{\partial t^2}$$

$$+ \beta_t \frac{\partial \phi}{\partial t} = T_e + \frac{1}{K' A_f G} \left[- EI_f \frac{\partial^2 T_e}{\partial z^2} + \rho I_f \frac{\partial^2 T_e}{\partial t^2} + \beta_w \frac{\partial T_e}{\partial t}\right] \qquad (6.16)$$

Similarly, eliminating ϕ between Eqs.(6.4) and (6.5) yields the uncoupled equation for ψ given by:

$$\left[\frac{EI_f C_s}{K' A_f} + EC_w\right] \frac{\partial^4 \psi}{\partial z^4} - \left[\frac{E \rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right] \frac{\partial^4 \psi}{\partial z^2 \partial t^2}$$

$$- GC_s \frac{\partial^2 \psi}{\partial z^2} - \left[\frac{EI_f \beta_t}{K' A_f G} + \frac{\beta_w C_s}{K' A_f} + \frac{\beta_w h^2}{2}\right] \frac{\partial^3 \psi}{\partial z^2 \partial t}$$

$$+ \frac{\rho^2 I_p I_f}{K' A_f G} \frac{\partial^4 \psi}{\partial t^4} + \left[\frac{\rho I_f \beta_t}{K' A_f G} + \frac{\rho I_p \beta_w}{K' A_f G}\right] \frac{\partial^3 \psi}{\partial t^3}$$

$$+ \left[\rho I_p + \frac{\beta_t \beta_w}{K' A_f G}\right] \frac{\partial^2 \psi}{\partial t^2} + \beta_t \frac{\partial \psi}{\partial t} = \frac{h}{2} \frac{\partial T_e}{\partial z} \qquad (6.17)$$

As expected, the left-hand sides of Eqs.(6.16) and (6.17) are identical.

Substituting Eqs.(6.6), (6.7), (6.12), (6.13) and (6.14) into Eqs.(6.16) and (6.17) results in:

$$\left[\frac{n^4\pi^4}{L^4}\left(\frac{EI_fC_s}{K'A_f}+EC_w\right)+\frac{n^2\pi^2GC_s}{L^2}\right]F_n(t)$$

$$+\left\{\beta_t+\frac{n^2\pi^2}{L^2}\left(\frac{EI_f\beta_t}{K'A_fG}+\frac{\beta_wC_s}{K'A_f}+\frac{\beta_wh^2}{2}\right)\right\}\dot{F}_n(t)$$

$$+\left\{\rho I_p+\frac{\beta_t\beta_w}{K'A_fG}+\frac{n^2\pi^2}{L^2}\left(\frac{E\rho I_pI_f}{K'A_fG}+\frac{C_s\rho I_f}{K'A_f}+\frac{\rho I_fh^2}{2}\right)\right\}\ddot{F}_n(t)$$

$$+\left(\frac{\rho I_f\beta_t}{K'A_fG}+\frac{\rho I_p\beta_w}{K'A_fG}\right)\dddot{F}_n(t)+\frac{\rho^2I_pI_f}{K'A_fG}\ddddot{F}_n(t)$$

$$=\left(1+\frac{n^2\pi^2EI_f}{K'A_fGL^2}\right)\tau_n(t)+\frac{\beta_w}{K'A_fG}\dot{\tau}_n(t)+\frac{\rho I_f}{K'A_fG}\ddot{\tau}_n(t) \qquad (6.18)$$

$$\left\{\frac{n^4\pi^4}{L^4}\left(\frac{EI_fC_s}{K'A_f}+EC_w\right)+\frac{n^2\pi^2GC_s}{L^2}\right\}G_n(t)$$

$$+\left\{\beta_t+\frac{n^2\pi^2}{L^2}\left(\frac{EI_f\beta_t}{K'A_fG}+\frac{\beta_wC_s}{K'A_f}+\frac{\beta_wh^2}{2}\right)\right\}\dot{G}_n(t)$$

$$+\left\{\rho I_p+\frac{\beta_t\beta_w}{K'A_fG}+\frac{n^2\pi^2}{L^2}\left(\frac{E\rho I_pI_f}{K'A_fG}+\frac{C_s\rho I_f}{K'A_f}+\frac{\rho I_fh^2}{2}\right)\right\}\ddot{G}_n(t)$$

$$+\left(\frac{\rho I_f\beta_t}{K'A_fG}+\frac{\rho I_p\beta_w}{K'A_fG}\right)\dddot{G}_n(t)+\frac{\rho^2I_pI_f}{K'A_fG}\ddddot{G}(t)=\frac{n\pi h}{2L}\tau_n(t) \qquad (6.19)$$

where dots denote differentiations with respect to time. Eqs.(6.18) and (6.19) contain an exciting torsional function $\tau_n(t)$ which can be of any form.

6.4. RESPONSE TO A UNIFORMLY DISTRIBUTED TORSIONAL FORCING FUNCTION SINUSOIDAL IN TIME:

For purposes of detailed numerical results, let $T_e(z,t)$ be

$$T_e(z,t) = T_o \sin \omega t \tag{6.20}$$

where T_o is a constant and ω the torsional excitation frequency. Then, from Eq.(6.15) it follows that:

$$\tau_n(t) = \frac{4T_o}{n\pi} \sin \omega t, \quad n = 1,3,5,\ldots \tag{6.21}$$

Assuming a solution in the form

$$F_n(t) = A_n \sin \omega t + B_n \cos \omega t \tag{6.22}$$

Substituting Eqs.(6.21) and (6.22) into Eq.(6.18), and equating coefficients of $\sin \omega t$ and $\cos \omega t$ yields

$$A_n = \frac{4\,T_o\left\{K_{1n}\left[K' A_f G + (n^2\pi^2/L^2)EI_f - \rho I_f \omega^2\right] + K_{2n}\beta_W \omega\right\}}{n\pi K' A_f G\,(K_{1n}^2 + K_{2n}^2)} \tag{6.23}$$

$$B_n = \frac{4\,T_o\left\{K_{1n}\beta_W \omega - K_{2n}\left[K' A_f G + (n^2\pi^2/L^2)EI_f - \rho I_f \omega^2\right]\right\}}{n\pi K' A_f G(K_{1n}^2 + K_{2n}^2)} \tag{6.24}$$

where

$$K_{1n} = \left\{ \left[\frac{n^4\pi^4}{L^4} \left(\frac{EI_f C_s}{K' A_f} + EC_w \right) + \frac{n^2\pi^2 GC_s}{L^2} \right] \right.$$

$$- \left[\rho I_p + \frac{\beta_t \beta_w}{K' A_f G} + \frac{n^2\pi^2}{L^2} \left(\frac{E \rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2} \right) \right] \omega^2$$

$$\left. + \frac{\rho^2 I_p I_f}{K' A_f G} \omega^4 \right\} \tag{6.25}$$

$$K_{2n} = \left\{ \omega\beta_t \left(1 + \frac{n^2\pi^2 EI_f}{K' A_f G L^2} \right) + \omega\beta_w \frac{n^2\pi^2}{L^2} \left(\frac{C_s}{K' A_f} + \frac{h^2}{2} \right) \right.$$

$$\left. - \frac{\omega^3 \rho}{K' A_f G} \left(\beta_t I_f + \beta_w I_p \right) \right\} \tag{6.26}$$

Similarly, assuming a solution

$$G_n(t) = C_n \sin\omega t + D_n \cos\omega t \tag{6.27}$$

and substituting Eq.(6.21) and (6.27) into Eq.(6.19) yields:

$$C_n = \frac{2 T_o h K_{1n}}{L(K_{1n}^2 + K_{2n}^2)} \; ; \quad D_n = \frac{-2 T_o h K_{2n}}{L(K_{1n}^2 + K_{2n}^2)} \tag{6.28}$$

where K_{1n} and K_{2n} are defined by Eqs.(6.25) and (6.26).

Of course, Eqs.(6.22) and (6.27) may be replaced in a more convenient phase angle form as:

$$F_n(t) = \sqrt{A_n^2 + B_n^2}\,\sin(\omega t + \arctan B_n/A_n) \qquad (6.29)$$

$$G_n(t) = \sqrt{C_n^2 + D_n^2}\,\cos(\omega t + \arctan D_n/C_n) \qquad (6.30)$$

Further we note that $D_n/C_n = - B_n/A_n$

6.5. FREE AND FORCED VIBRATIONS OF A CLASSIC BEAM SIMPLY SUPPORTED AT BOTH ENDS:

For purposes of comparing with the preceding results, let us now summarize the classic solution. In the case of the classic beam based on Timoshenko torsion theory, the effects of longitudinal inertia and shear deformation are neglected and by putting $1/K' = 0$ and $\rho I_f = 0$ in Eq.(6.16) we obtain:

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} + \beta_t \frac{\partial \phi}{\partial t} = T_e \qquad (6.31)$$

Considering first, free vibrations with no damping, the differential equation becomes

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - GC_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (6.32)$$

which was treated in detail by Gere (32).

The solution to this equation in terms of circular and hyperbolic functions is well known (32). It can be seen that a function which satisfies the boundary conditions of a beam simply supported at both ends is given by:

$$\phi = \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi z}{L} \qquad (6.33)$$

Substituting Eq.(6.33) into Eq.(6.32) and recognizing that the resulting equation must be satisfied for all values of z within $0 \leq z \leq L$ gives

$$\rho I_p \ddot{F}_n(t) + \frac{n^2\pi^2}{L^2} \left(\frac{n^2\pi^2 EC_w}{L^2} + GC_s\right) F_n(t) = 0 \tag{6.34}$$

From Eq.(6.34), the well known (32) frequency equation is found to be:

$$p_n = \frac{n\pi}{\sqrt{2}L} \left[\frac{n^2\pi^2 EC_w + L^2 GC_s}{I_p L^2}\right]^{1/2} \tag{6.35}$$

For the steady-state solution of the forced, damped vibration problem as before, assume

$$\phi = \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi z}{L} \tag{6.36}$$

$$T_e(z,t) = \sum_{n=1}^{\infty} \tau_n(t) \sin \frac{n\pi z}{L} \tag{6.37}$$

where, from Eq.(6.15)

$$\tau_n(t) = \frac{4T_o}{n\pi} \sin \omega t, \quad (n=1,3,5,\ldots) \tag{6.38}$$

Substituting Eqs.(6.36), (6.37) and (6.38) into Eq.(6.31) yields

$$\frac{n^2\pi^2}{L^2} \left[\frac{n^2\pi^2}{L^2} EC_w + GC_s\right] F_n(t) + \beta_t \dot{F}_n(t) + \rho I_p \ddot{F}_n(t) = \frac{4T_o}{n\pi} \sin \omega t \tag{6.39}$$

having a steady-state solution

$$F_n(t) = E_n \sin \omega t + H_n \cos \omega t \tag{6.40}$$

Substituting Eq.(6.40) into Eq.(6.39), we obtain

$$E_n = \frac{(4T_o/n\pi)\left\{(n^2\pi^2/L^2)\left[(n^2\pi^2/L^2)EC_w + GC_s\right] - \omega^2 \rho I_p\right\}}{\left\{(n^2\pi^2/L^2)\left[(n^2\pi^2/L^2)EC_w + GC_s\right] - \omega^2 \rho I_p\right\}^2 + (\beta_t \omega)^2} \quad (6.41)$$

$$H_n = \frac{-(4T_o\ \beta_t \omega/n\pi)}{(n^2\pi^2/L^2)\left[(n^2\pi^2/L^2)EC_w + GC_s\right] - \omega^2 \rho\ I_p\Big\}^2 + (\beta_t \omega)^2} \quad (6.42)$$

or

$$F_n(t) = \frac{4T_o}{n\pi}\left\{\rho^2 I_p^2 (p_n^2 - \omega^2)^2 + (\beta_t \omega)^2\right\}^{1/2} \sin(\omega t + \theta) \quad (6.43)$$

where

$$\tan\theta = \frac{-\beta_t \omega}{\rho I_p (p_n^2 - \omega^2)} \quad (6.44)$$

6.6. <u>DISCUSSION OF NUMERICAL RESULTS</u>:

The solutions obtained were programmed on IBM-1130 Computer at Andhra University, Waltair, to allow a numerical study of the effects of the parameters involved. Some of the interesting results obtained are shown in Figs.6.2 to 6.8. In Figs.6.2 to 6.8, only the response of the first mode shape is considered. The values of the constants used for these figures are as follows:

$n=1$; $\rho = 0.00884332 (lbs/in^3)$; $E = 30 \times 10^6\ (lbs/in^2)$;

$G = 12 \times 10^6 (lbs/in^2)$; $A_f = 20.7584 (in^2)$; $I_f = 469.532 (in^4)$;

$I_p = 17245.7 (in^4)$; $C_s = 27.3252 (in^4)$; $C_w = 3{,}02{,}231 (in^6)$;

$L = 760 (in)$ and $T_o = 1.0$,

which correspond to a wide-flanged steel I-beam, 36 WF 230, with width of the flanges b = 16.475(in), height between the center lines of the flanges h = 35.88(in), thickness of the web t =0.765 (in) and thickness of the flanges t_f= 1.26(in).

Fig.6.2 is the plot of torsional amplitude against forcing function frequency with varying values of torsional damping for the classical beam based on Timoshenko torsion theory.

Figs.6.3, 6.4 and 6.5 are the plots of amplitude versus frequency including the effects of longitudinal inertia and shear deformation. For each set of the curves, the value of β_w, the damping associated with warping angle, is held constant while the values of torsional damping β_t are varied.

It can be observed that the general shapes of the plots do not differ at all from that of Fig.6.2, i.e., shear deformation and longitudinal inertia effects do not radically alter the form of the amplitude-frequency curves. As expected, increasing the damping associated with warping angle has the effect of lowering the amplitudes.

Figs.6.6, 6.7 and 6.8 are also amplitude frequency plots including longitudinal inertia and shear deformation effects, but for each set of curves β_t is held constant while β_w is varied from zero to 10^5. Again, the general form of the curves is not unlike that for the classical beam. However, comparing Figs.6.6, 6.7 and 6.8 with Figs.6.3, 6.4 and 6.5, it will be readily seen that the variation of damping associated with angle of twist β_t, has a much stronger influence on the curves than the variation

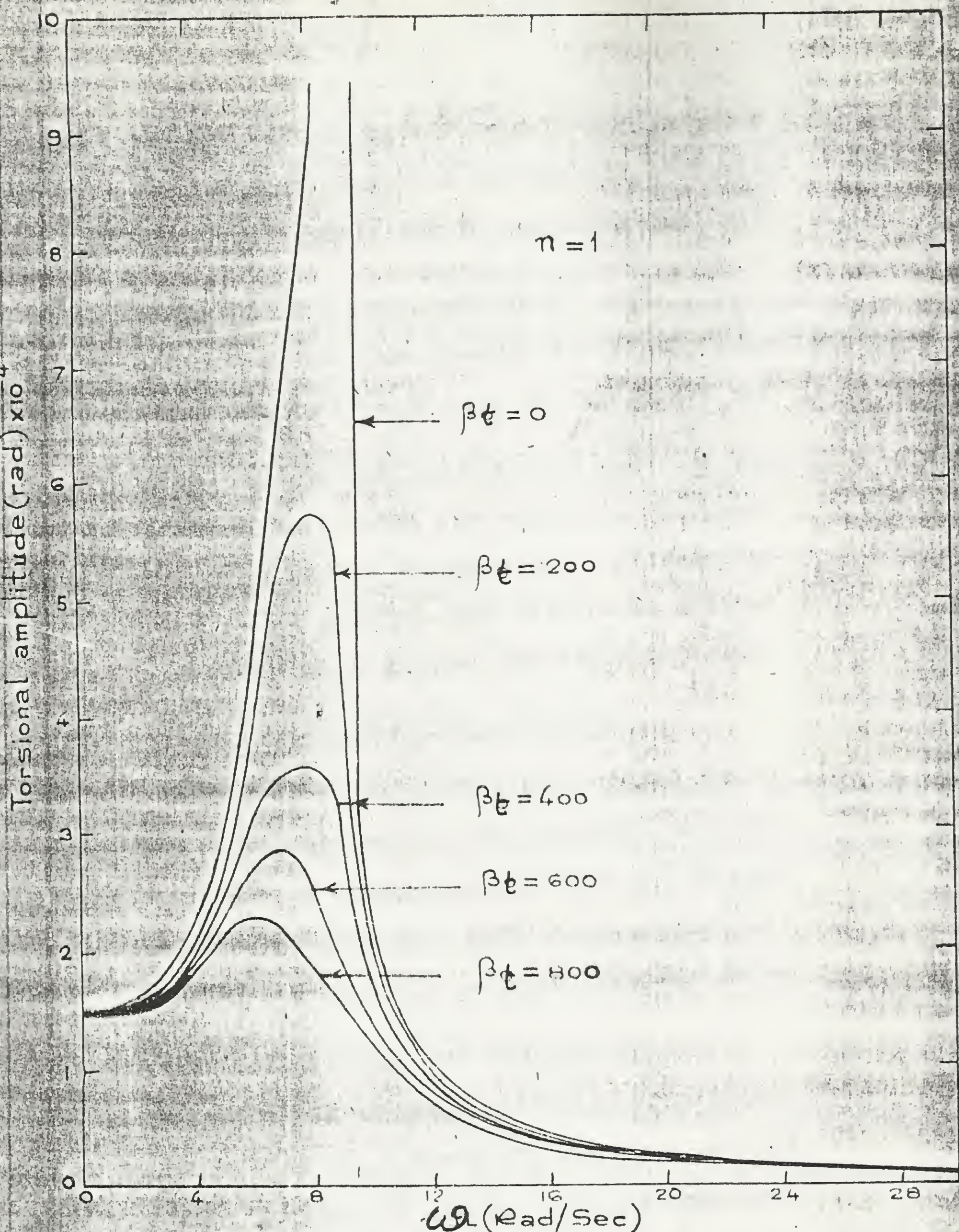

Fig. 6.2. Classic beam Timoshenko torsion theory.

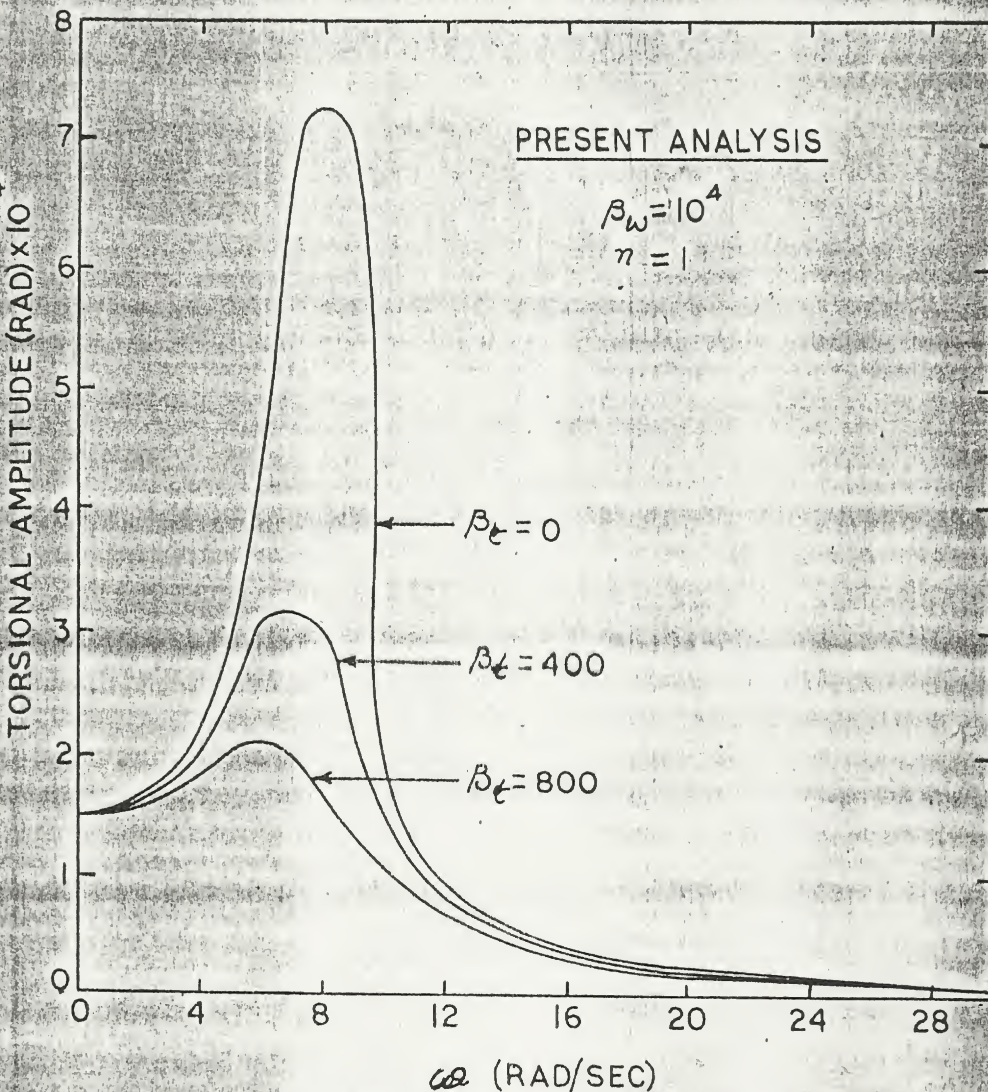

Fig. 6.3. Present analysis.

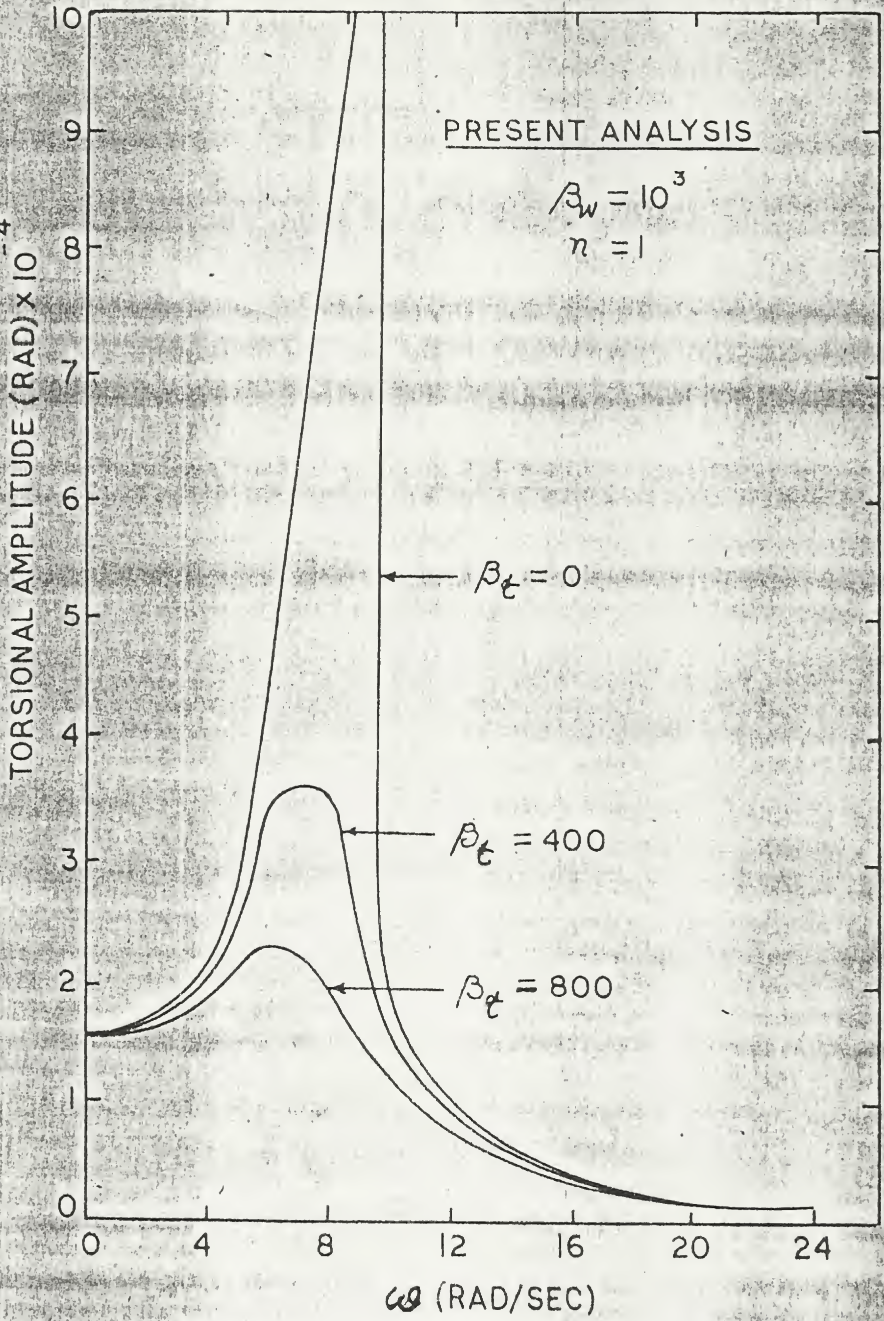

Fig. 6.4. Present analysis.

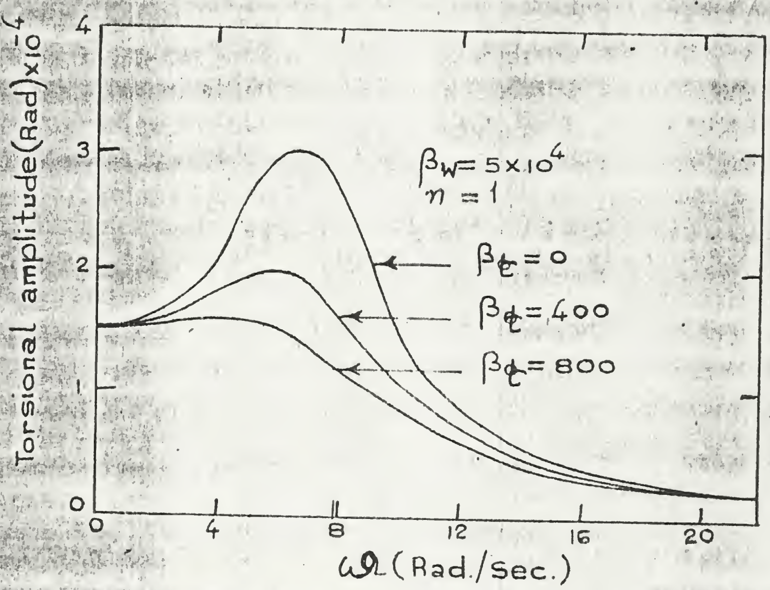

Fig. 6.5. Present analysis.

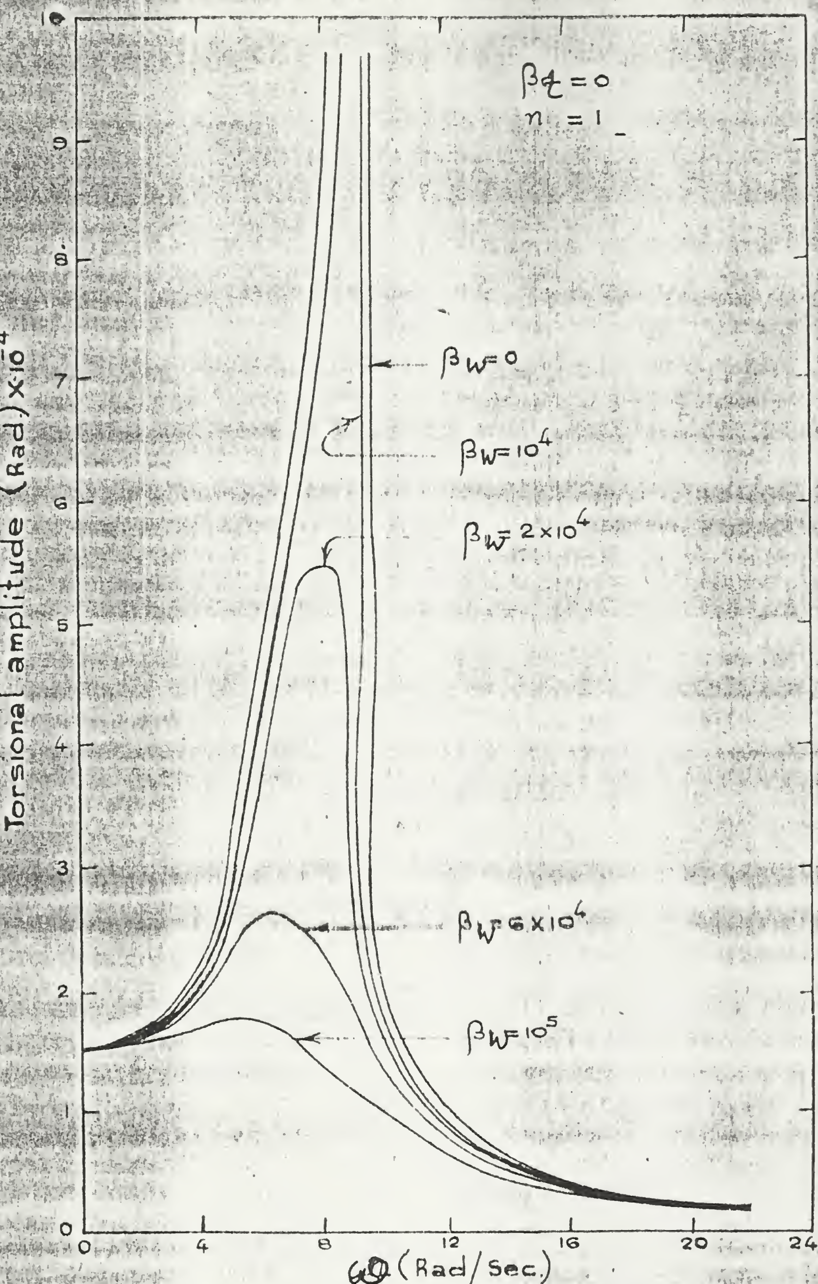

Fig 6.6. Present analysis.

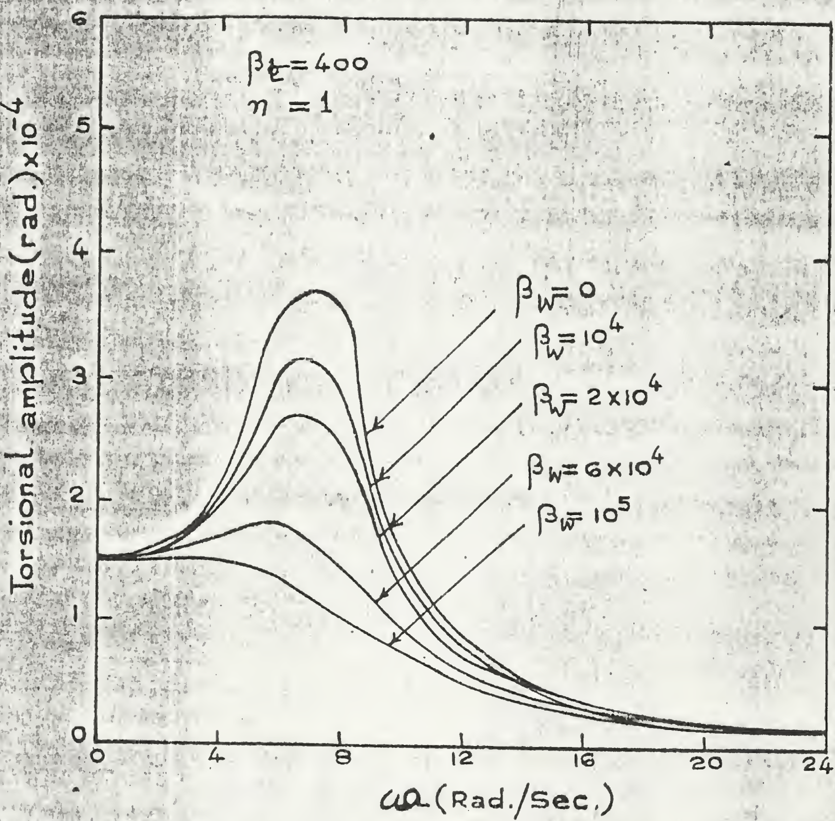

Fig. 6.7. Present Analysis.

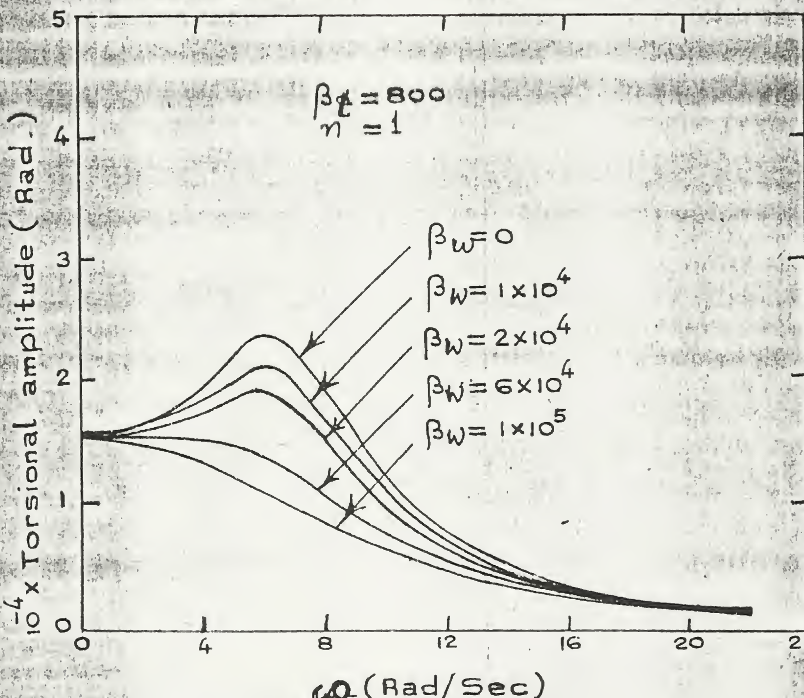

Fig. 6.8. Present analysis.

TABLE - 6.1

Values of the natural frequencies and maximum total torsional amplitudes for various modes of vibration of a simply supported beam.

Mode Number n	Natural Frequency		Maximum Total Amplitude	
	Classic Beam	Present Analysis	Classic Beam	Present Analysis
1	245.211	235.791	1.38790×10^{-7}	1.47853×10^{-7}
3	2,171.970	1,662.560	5.89665×10^{-10}	9.47434×10^{-10}
5	6,025.440	3,558.770	4.59715×10^{-11}	12.36510×10^{-11}
7	11,805.600	5,539.010	8.55382×10^{-12}	36.90330×10^{-12}
9	19,512.500	7,515.080	2.43537×10^{-12}	15.78190×10^{-12}

of damping associated with warping angle β_w. Therefore, including the effects of longitudinal inertia and shear deformation, the torsional velocity damping is more significant than the warping-velocity damping.

Further, to consider the effects on higher modes, light torsional damping, (β_t=200, β_w=0) will be applied to a beam of large depth to length ratio. Keeping the same physical parameters as above, except letting L = 100 (in) to emphasize the shear deformation effects, the 'maximum total torsional amplitude' response may be computed. This is the maximum torsional amplitude obtained due to superposition of the responses of all modes when the separate natural frequencies are successively ex cited. Maximum total torsional amplitudes are given in Table 6.1, for the first nine symmetric mode shapes of the simply supported beam. From Table 6.1, it is observed that as the mode number n increases the difference between the natural frequencies of the classical beam and, those obtained from the present analysis including the effects of longitudinal inertia and shear deformation, also increases. As shown in Chapters IV and V, the natural frequencies obtained by including the effects of longitudinal inertia and shear deformation are lower than those for the classic beam. However, the amplitudes obtained including longitudinal inertia and shear deformation are larger than those for the classic beam.

CHAPTER - VII

TORSIONAL WAVE PROPAGATION IN ORTHOTROPIC THIN-WALLED BEAMS OF OPEN SECTION INCLUDING THE EFFECTS OF LONGITUDINAL INERTIA AND SHEAR DEFORMATION*.

7.1. INTRODUCTION:

In the previous Chapters, free and forced torsional vibrations of short thin-walled beams of open section including the effects of longitudinal inertia and shear deformation are analyzed both by exact and approximate methods. The present Chapter deals with the important problem of torsional wave propagation in orthotropic thin-walled beams of open section including the second order effects.

Though there exists a good amount of work on the analysis of flexural wave propagation, comparable torsional wave analysis was virtually neglected and very few papers on this topic have been published. The reason is the fact that Coulomb theory gives the same first-mode results as the exact theory. The available information is almost limited to the circular cylindrical bars. Thus, there exists a lack of satisfactory approximate and exact theories for torsional wave propagation in non-circular bars, especially those used in structural applications such as thin-walled beams of open section.

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* A paper by the author based on the results of this Chapter is accepted for publication in the Journal of the Aeronautical Society of India. See Ref.(54).

An inadequacy of St.Venant's classical torsion theory for short wave lengths was hinted at by Love (76), who suggested a correction for the longitudinal inertia associated with torsional deflection. Vlasov (107) also introduced the effect of longitudinal inertia in his torsional analysis of thin-walled beams. However, both the elementary theory and Love's or Vlasov's approximation have the same defects as do their counterparts in longitudinal wave-propagation theory. The dynamic equation used by Gere (32) in his torsion analysis was essentially that previously derived by Timoshenko (98) and included the effect of warping of the cross section. These equations are found to lead to physically absurd results for short wavelengths. Aggarwal and Cranch (4) presented a strength of materials theory including the effects of warping of the cross section, longitudinal inertia and shear deformation. This theory was found to lead to theoretically satisfactory results for the first mode of transmission over a wavelength spectrum which included moderately short wavelengths, and that it agreed with previous approximations for large wavelengths. The group velocity for the second mode was found to increase monotonically from zero for the longest waves to the bar velocity for very short wavelengths. This was in agreement in form with the higher modes of the exact theory for circular cylindrical bars (88,25).

All the above work, and a ~~host~~ number of other investigations involving torsional wave propagation phenomena in thin-walled beams, concerns isotropic materials. Anisotropic materials have not been approached to the best of author's knowledge. As is well known, anisotropy of the material introduced considerable complications in the computational part of the solution.

The present Chapter therefore, aims at investigating the problem of torsional wave propagation in orthotropic thin-walled beams of open section including the effects of longitudinal inertia and shear deformation, from the strength of materials approach. This approach is attractive for its physical directness. More specifically, the interest is to find what values of the wave frequency result from the elementary theory established for the anisotropic analog of the isotropic thin-walled beams of open section including the effects of longitudinal inertia and shear deformation. To this end, the equation of motion for free torsional vibrations of thin-walled beams of open section of orthotropic material including the second order effects is established, analogous to that for isotropic material. It is shown herein that, for some anisotropic materials, the corrections due to longitudinal inertia and shear deformation may be of one order of magnitude greater than the correction in the isotropic case. Graphs are also given for the phase velocity versus inverse wavelength for various aspect ratios of beams of different materials.

7.2. ANALYSIS AND EXAMPLES:

For definiteness and simplicity, let us take the material of the thin-walled open section beam to be orthotropic, with one axis of elastic symmetry, z-axis, directed along the axis of the beam.

As is well known the fundamental equation of elementary theory of flange-bending retains its validity for anisotropic materials of the most general type, provided the isotropic Young's modulus is replaced by the modulus E_{zz} for extention-compression along the axis of the bar.

In symbols,

$$M = E_{zz} I_f \frac{\partial \psi}{\partial z} \qquad (7.1)$$

analagous to the Eq.(4.4) for the isotropic beams.

Now, in the derivation, in strength of materials, of the formula for the maximum shear stress in flange-bending,

$$\tau_{zx}(\max) = -\frac{QS_o}{I_f t_w}, \qquad (7.2)$$

no specific elastic properties of the material besides certain, symmetric conditions, are postulated. This equation, therefore, is certainly valid (in the same sense of strength of materials) for the elastic symmetrices involved in the orthotropic thin-walled open section beam characterized earlier. For such a beam, with G_{zx} as the pertinent shear modulus,

$$\tau_{zx} = G_{zx} \epsilon_{sh} \qquad (7.3)$$

so that

$$-Q = K' A_f G_{zx} \epsilon_{sh} \qquad (7.4)$$

where ϵ_{sh} is the shear strain at the center of the flange, x=0, given by

$$\epsilon_{sh} = \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) \tag{7.5}$$

In Eq.(7.2) all others being previously defined, S_o stands for the statical moment with respect to neutral axis. In Eq.(7.4) K' is the shear coefficient which depends upon the shape of the cross section and is given by

$$K' = \frac{I_f t_w}{S_o A_f} . \tag{7.6}$$

There is no difference between Eqs.(7.1) and (7.4) and the corresponding equations in the isotropic case i.e., Eqs.(4.4) and (4.7) of Chapter IV, except for the modulii E_{zz} and G_{zx} standing for E and G. One can therefore avoid all the transformation and proceed directly to derive the frequency equation.

Following the procedure in Chapter IV, the equations of motion can be now written for torsional vibrations of orthotropic thin-walled beams of open section as:

$$G_{zx} C_s \frac{\partial^2 \phi}{\partial z^2} + K' A_f G_{zx} h \left(\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z} \right) = \rho I_p \frac{\partial^2 \phi}{\partial t^2} \tag{7.7}$$

and

$$K' A_f G_{zx} \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right) + E_{zz} I_f \frac{\partial^2 \psi}{\partial z^2} = \rho I_f \frac{\partial^2 \psi}{\partial t^2} \tag{7.8}$$

Eliminating ψ between Eqs.(7.7) and (7.8) a single equation in ϕ may be obtained as:

$$\left[\frac{E_{zz} I_f C_s}{K' A_f G_{zx}} + E_{zz} C_w\right] \frac{\partial^4 \phi}{\partial z^4} - \left[\frac{\rho E_{zz} I_p I_f}{K' A_f G_{zx}} + \frac{\rho C_s I_f}{K' A_f} + \frac{\rho I_f h^2}{2}\right] \frac{\partial^4 \phi}{\partial z^2 \partial t^2}$$

$$- G_{zx} C_s \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} + \frac{\rho^2 I_p I_f}{K' A_f G_{zx}} \frac{\partial^4 \phi}{\partial t^4} = 0 \qquad (7.9)$$

For a wave-form solution in long beams, consider a sinusoidal wave,

$$\phi \sim e^{i\delta_1 (z - C_p t)} \qquad (7.10)$$

propagating along the beam. In Eq.(7.10), δ_1 is the wave number $= 2\pi/\Lambda$, Λ being the wavelength, C_p the phase velocity for torsional waves, and t is the time.

Substituting ϕ from Eq.(7.10) into Eq.(7.9), the frequency equation for torsional waves is obtained as

$$\frac{\rho I_f}{K'} \left(\frac{C_p}{C_2}\right)^4 - \left[\frac{\rho I_f}{K'}\left(\frac{E_{zz}}{G_{zx}}\right) + \frac{\rho I_f}{I_p}\left(\frac{C_s}{K'} + \frac{A_f h^2}{2}\right) + \frac{\rho A_f}{\delta_1^2}\right] \left(\frac{C_p}{C_2}\right)^2$$

$$+ \left[\frac{\rho I_f}{I_p}\left(\frac{E_{zz}}{G_{zx}}\right)\left(\frac{C_s}{K'} + \frac{A_f h^2}{2}\right) + \frac{\rho A_f C_s}{I_p \delta_1^2}\right] = 0 \qquad (7.11)$$

where $C_2 = (G_{zx}/\rho)^{1/2}$ is the shear wave velocity. Eq.(7.11) determines the phase velocities of the torsional wave propagation in an orthotropic thin-walled open section beam.

Two cases of interest can be deduced from Eq.(7.11) as follows:

(1) Neglecting shear deformation, by letting $K' \to \infty$, the frequency Eq.(7.11) reduces to:

$$\left(\frac{C_p}{C_2}\right)^2 = \frac{C_s + 2\pi^2 (E_{zz}/G_{zx})\, I_f (h/\Lambda)^2}{I_p + 2\pi^2\, I_f (h/\Lambda)^2} \qquad (7.12)$$

Eq.(7.12) therefore is the frequency equation which includes the warping and longitudinal inertia effects of the cross section.

(2) Neglecting longitudinal inertia and shear deformation, by letting $\rho I_f = 0$, $K' \to \infty$, the frequency equation (7.11) reduces to:

$$\left(\frac{C_p}{C_2}\right)^2 = \frac{1}{I_p}\left[C_s + 2\pi^2\, I_f (E_{zz}/G_{zx})(h/\Lambda)^2 \right] \qquad (7.13)$$

which is the frequency equation including the effect of warping only and represents the Timoshenko tortion theory (32).

Returning now to the general Eq.(7.11) which includes both the second order effects, it may written in an alternative form as:

$$\left(\frac{C_p}{C_2}\right)^4 - \left[\bar{\alpha}_3 + \bar{\beta}_3 + \frac{\bar{\eta}_5}{4\pi^2}\left(\frac{\Lambda}{h}\right)^2 \right]\left(\frac{C_p}{C_2}\right)^2$$

$$+ \left[\bar{\alpha}_3\bar{\beta}_3 + \frac{\bar{\eta}_5\bar{\xi}_2}{4\pi^2}\left(\frac{\Lambda}{h}\right)^2 \right] = 0 \qquad (7.14)$$

where

$$\bar{\alpha}_3 = E_{zz}/G_{zx} \qquad (7.15)$$

$$\bar{\beta}_3 = \frac{1}{I_p}\left[C_s + (1/2)\, K' A_f h^2 \right] \qquad (7.16)$$

$$\bar{\eta}_{\zeta} = K' A_f h^2/I_f \tag{7.17}$$

and

$$\bar{\xi}_2 = C_s/I_p \tag{7.18}$$

Eq.(7.14) gives rise to to two modes of wave transmission. The new mode can be explained to arise from the coupled interaction of the torsional deformation with the bending effects of shear deformation and longitudinal inertia. The phase velocities for the two modes are given by Eq.(7.14) as:

$$\left(\frac{C_p}{C_2}\right)^2 = \frac{1}{2}\left\{ \left[\bar{\alpha}_3 + \bar{\beta}_3 + \frac{\bar{\eta}_{\zeta}}{4\pi^2}\left(\frac{\Lambda}{h}\right)^2\right] \pm \left[\left[\bar{\alpha}_3+\bar{\beta}_3+\frac{\bar{\eta}_{\zeta}}{4\pi^2}\left(\frac{\Lambda}{h}\right)^2\right]^2 - 4\left[\bar{\alpha}_3\bar{\beta}_3+\frac{\eta_{\zeta}\bar{\xi}_2}{4\pi^2}\left(\frac{\Lambda}{h}\right)^2\right]\right]^{1/2}\right\} \tag{7.19}$$

where the minus sign is taken for the first mode.

Eq.(7.19) defines the phase velocity as a function of the shape of the cross section. At very large wave lengths the results for the lower mode obtained from Eq.(7.19) will agree with those from previous theories. This is obvious because the deformation associated with long wave lengths is primarily that of rotation of the cross section with essentially no warping, no shear deformation and hence no dispersion. The improved theory due to Aggarwal and Cranch (4) displays finite wave velocity $C_2 \sqrt{\beta_3}$ for very short wavelengths as against the infinite wave velocities predicted by Timoshenko torsion theory and low wave velocities predicted by Saint-Venant torsion theory.

From Eq.(7.16) which defines β_3, it may be observed that for short wave lengths, the torsional stiffness effect is very small and the shear distortion of the flanges contributes more. The present analysis gives satisfactory results for wave lengths $\Lambda > t_w$ for the first mode and this coincides in the second mode with the form of the exact theory for circular cylindrical bars. The range of applicability of the first mode, $\Lambda > t_w$, gives a wave length spectrum which includes moderately short waves and high frequencies, and as such covers a range of practical interest. As an example, for the beam for which $b/h = 0.75$, $t_f/h = 0.050$ and $t_w/h = 0.040$ the theory is valid for wave lengths $h/\Lambda < 25$.

Despite the fact that Eq.(7.19) has a form identical with that given by Aggarwal and Cranch (4) for isotropic beams, there is a basic difference between the two equations. It consists in that, for isotropic bodies, the value of poisson's ratio ranges (at least in principle) from 0 to 0.5, so that the value of E/G in Eq.(7.19) falls between 2 and 3. On the other hand for anisotropic materials the values of E_{zz}/G_{zx} may be one and possibly even two orders of magnitude higher. So much so, both the corrections due to shear deformation, and the corrections for longitudinal inertia and shear deformation together, may become several times greater for anisotropic beams than they are for isotropic beams.

Table 7.1. Values of $\bar{\alpha}_3$ for various materials.

Material	$\bar{\alpha}_3 = E_{zz}/G_{zx}$
Isotropy	2.6
Orthotropy II	13.9
Orthotropy I	17.1
Transverse Isotropy	35.0
	(Average of the range 20 - 50)

The values of $\alpha_3 (= E_{zz}/G_{zx})$ for three types of anisotropic materials considered in this Chapter are given in Table 7.1. For an isotropic material the value of α is taken as 2.6.

7.3. RESULTS AND DISCUSSION:

Figs.7.1 to 7.8 show, the phase velocities for torsional waves in four wide-flanged I-beams which cover the practical range, having dimensions such as:

(1) $b/h=0.25$, $t_f/h=0.025$, $t_w/h=0.020$ (Figs.7.1 and 7.2)
(2) $b/h=0.50$, $t_f/h=0.040$, $t_w/h=0.025$ (Figs.7.3 and 7.4)
(3) $b/h=0.75$, $t_f/h=0.050$, $t_w/h=0.040$ (Figs.7.5 and 7.6)
(4) $b/h=1.00$, $t_f/h=0.10$, $t_w/h=0.050$ (Figs.7.7 and 7.8)

Of isotropic and three types of anisotropic materials having values of $\bar{\alpha}_3$, 2.6 (isotropic), 13.9 (orthotropy II), 17.1 (orthotropy I) and 35.0 (transverse isotropy). Figs.7.1, 7.3, 7.5 and 7.7 gives the results corresponding to the first mode for various values of

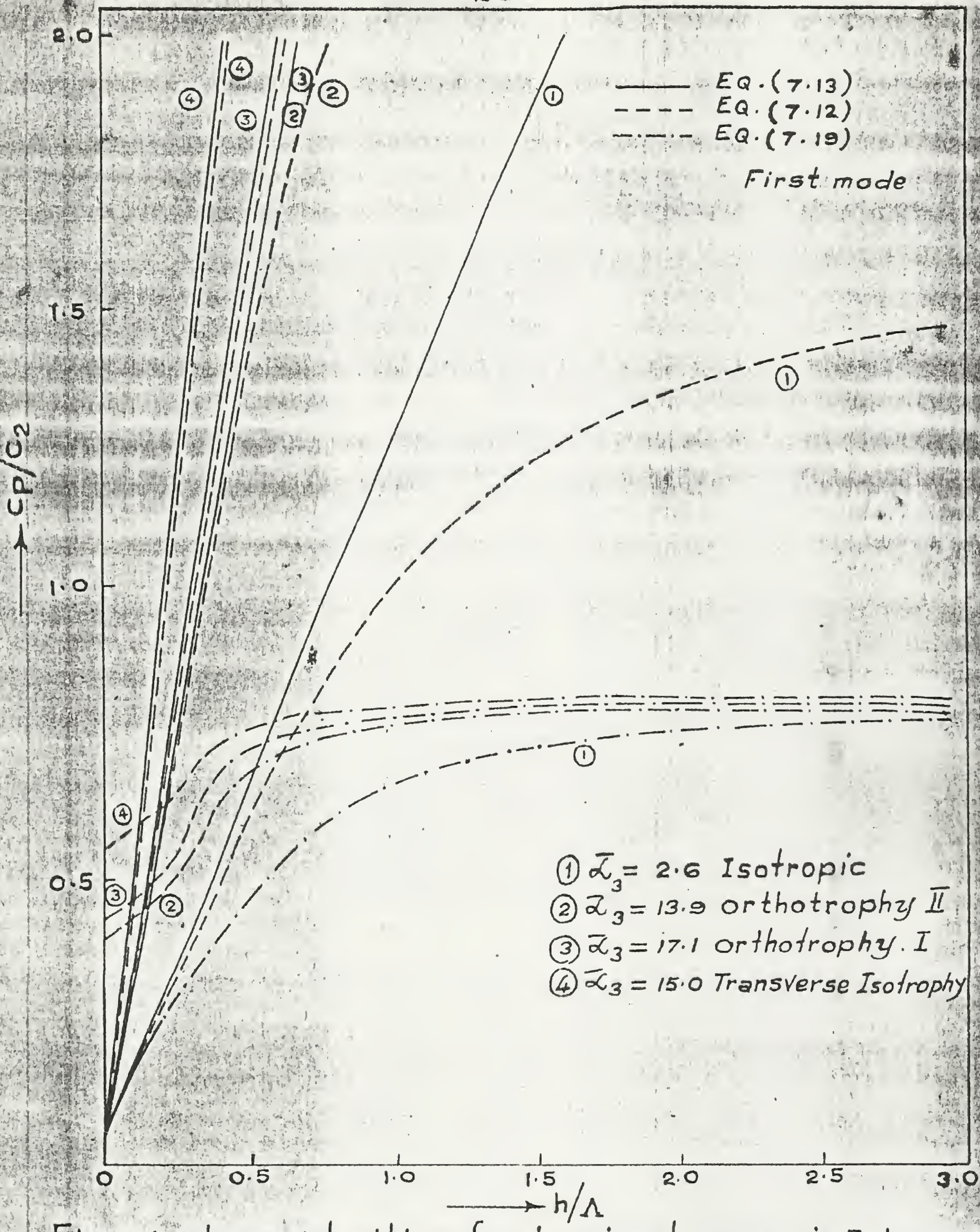

Fig. 7.1. phase velocities for torsional waves in I-beams

[$b/h = 0.50$; $t_f/h = 0.040$; $t_w/h = 0.025$]

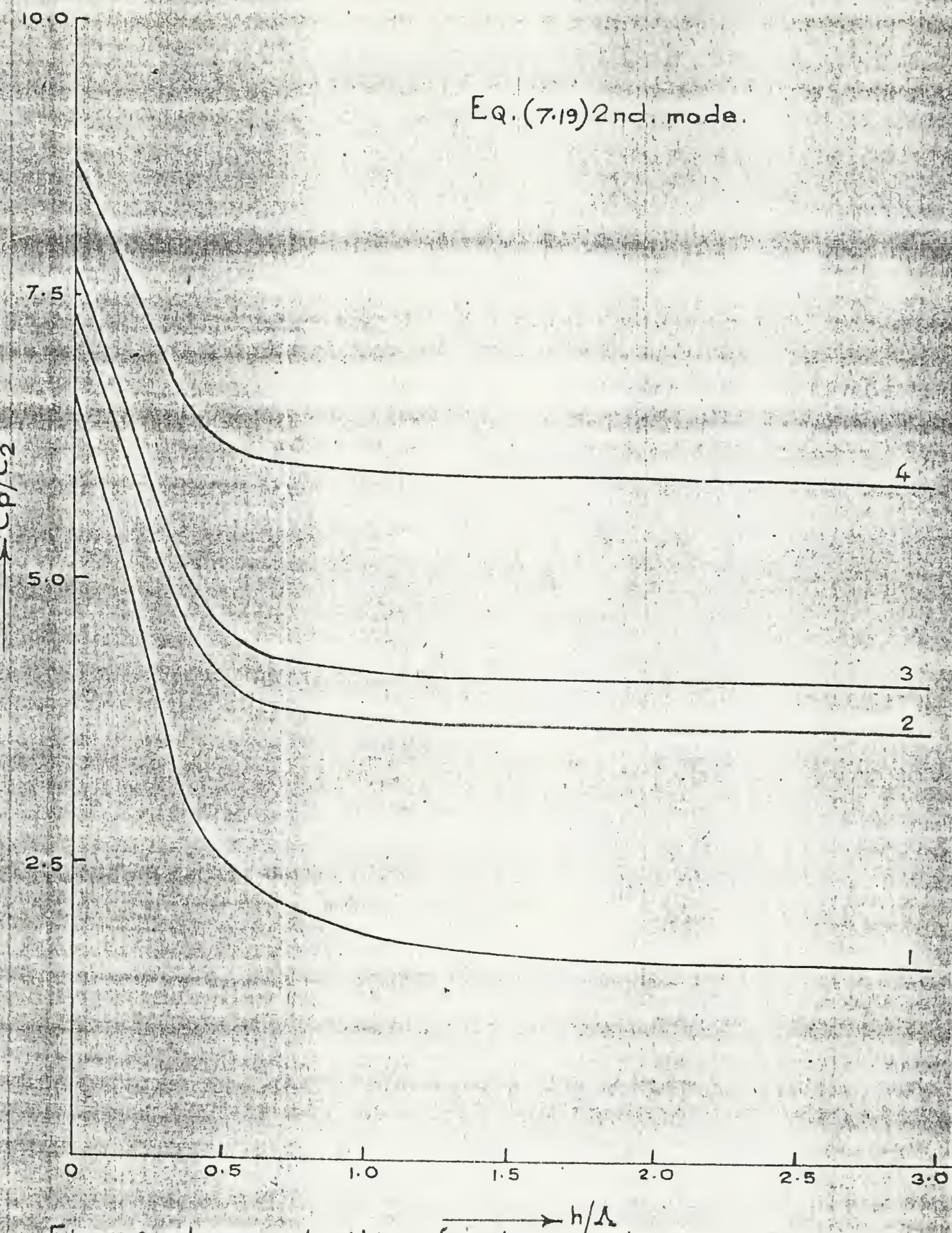

Fig. 7.2. phase velocities for torsional wave in I-beams.
[b/h = 0.50; $t_f/h = 0.040$; $t_w/h = 0.025$]

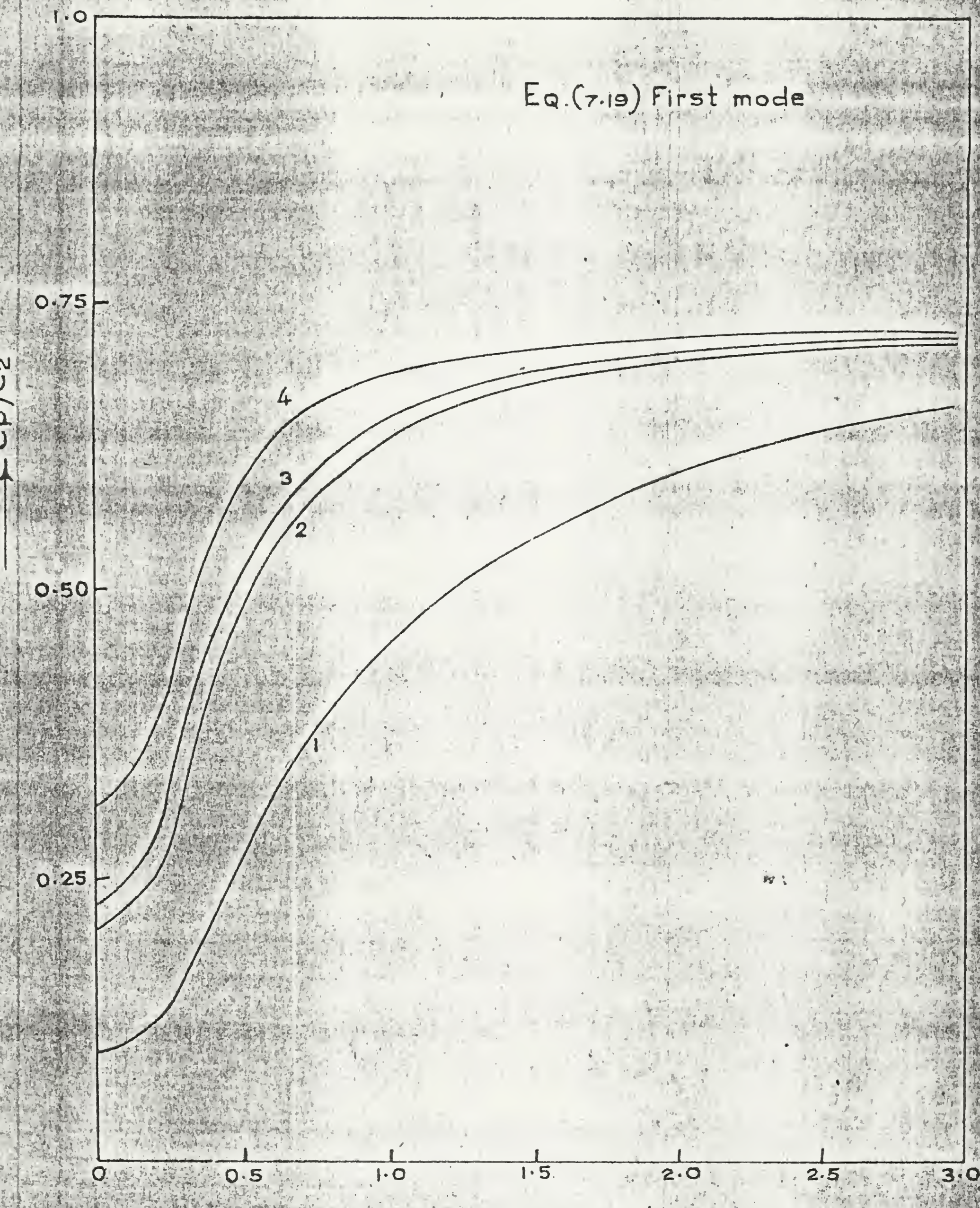

Fig.7.3. Phase velocities for torsional waves in I-beams.
[b/h = 0.25 ; t_f/h = 0.025 ; t_w/h = 0.020]

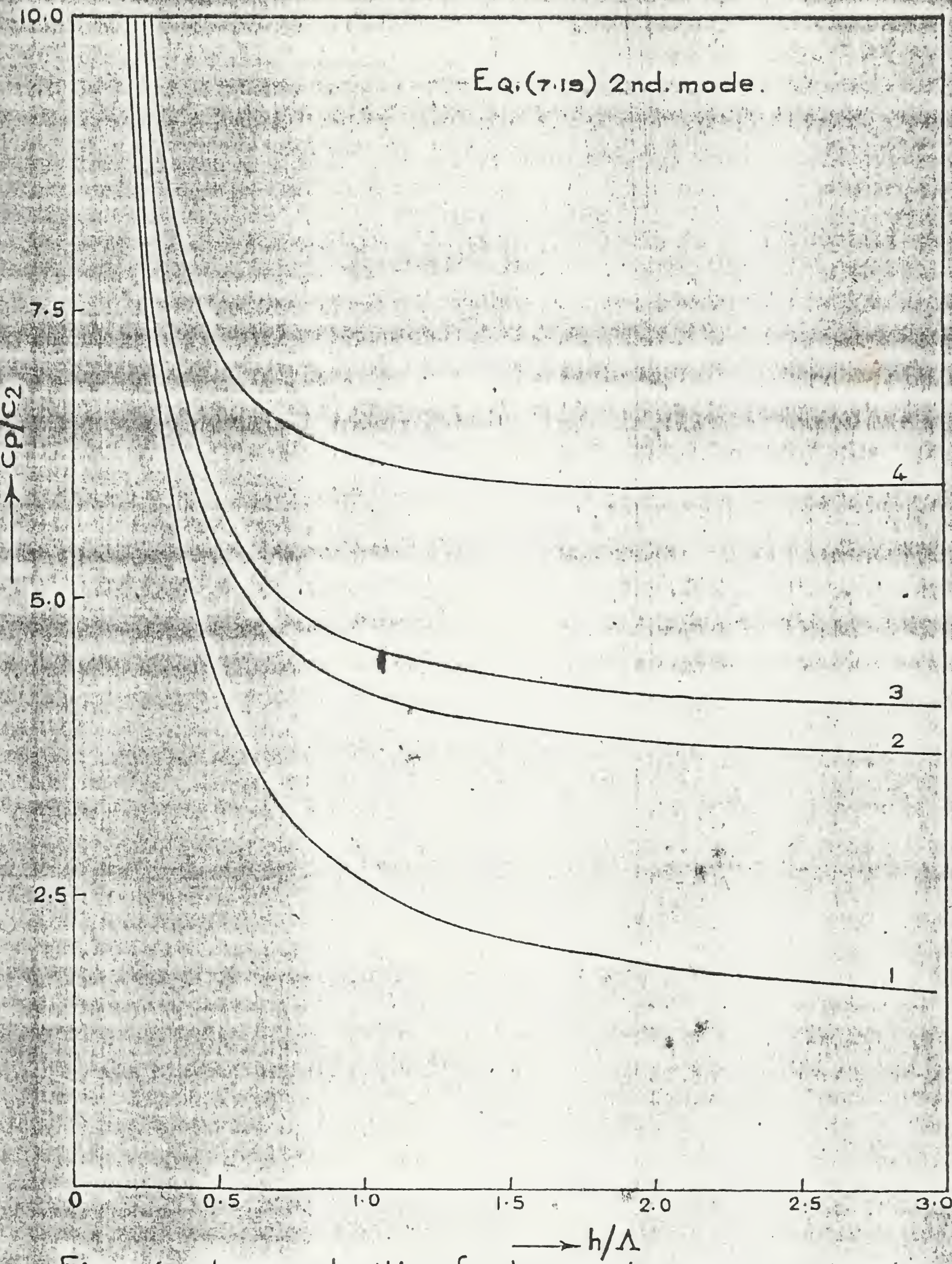

Fig. 7.4. Phase velocities for torsional waves in I-beams.
[$b_f/h = 0.25$; $t_f/h = 0.025$; $t_w/h = 0.020$]

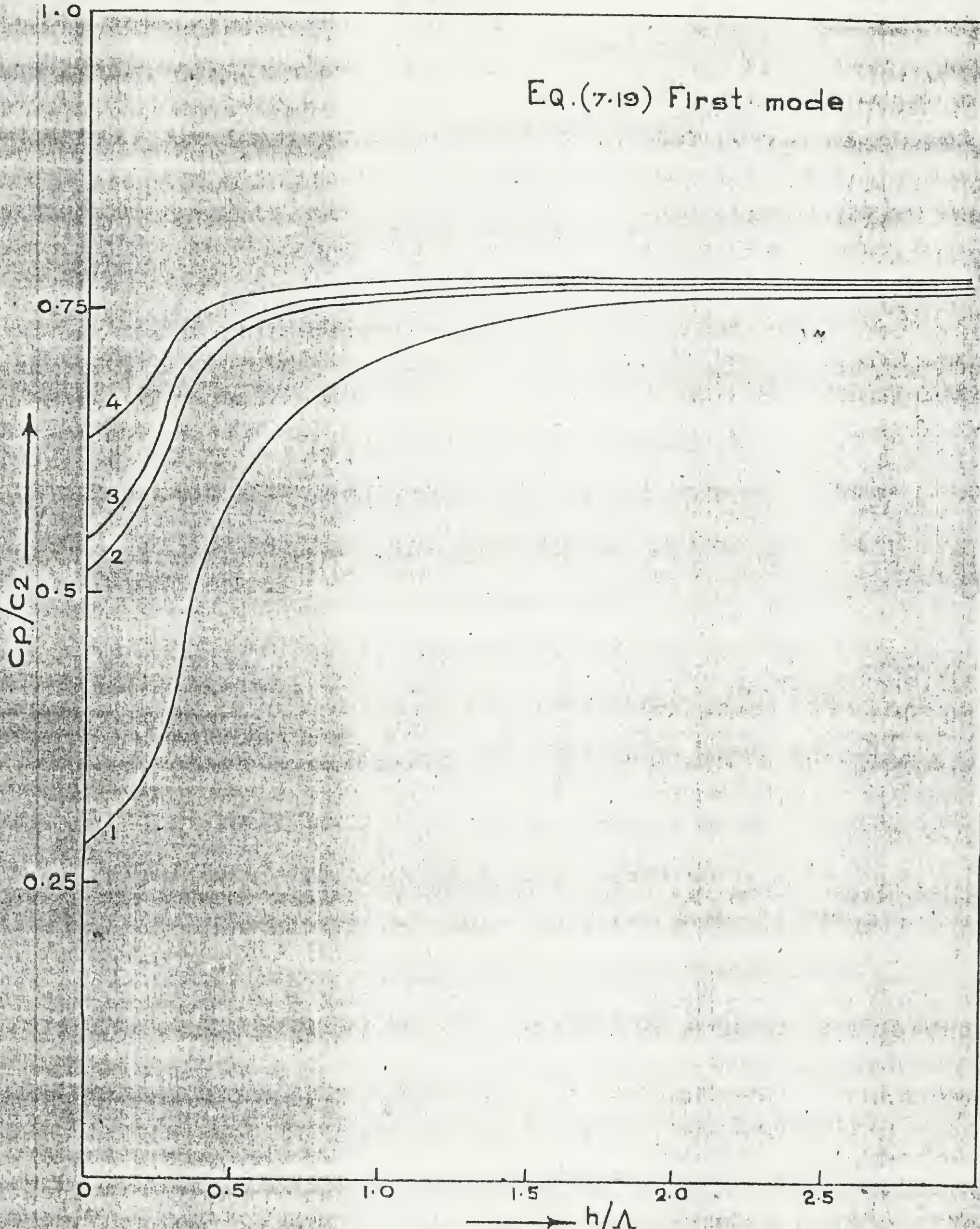

Fig. 7.5. Phase velocities for torsional waves in I-beams.
[$b/h = 0.75$; $t_f/h = 0.050$; $t_w/h = 0.040$]

α_3 for the four beams.

In drawing the graphs, the value of K' was taken as $\pi^2/12$. The phase velocities corresponding to the second mode for all values of α_3 can be observed, from Figs.7.2, 7.4, 7.6 and 7.8 for the four beams considered here, to decrease from infinite values for the longest waves to the beam velocity for the shortest waves.

The results for phase velocities obtained from Timoshenko torsion theory (Eq.7.13), the theory including warping and longitudinal inertia (Eq.7.12), and the theory including warping, longitudinal inertia and shear deformation (Eq.7.19) are compared in Fig.7.1 for beam (1) defined above, for the four values of $\bar{\alpha}_3$ considered in this work. In all cases the values of the phase velocities increase with increasing values of $\bar{\alpha}_3$.

From Fig.7.1, it can be observed that, at lower values of h/Λ, the phase velocities from Eq.(7.19), increase considerably with increasing values of $\bar{\alpha}_3$, but differ only slightly for different values of α at higher values of h/Λ. The values obtained from Eqs.(7.12) and (7.13) differ greatly at lower values of $\bar{\alpha}_3 (= 2.6)$ but differ slightly for higher values of $\bar{\alpha}_3$. Because of the above, it can be seen, that the percentage of influence of both longitudinal and shear deformation on the torsional wave propagation may increase drastically for increasing values of $\bar{\alpha}_3$ i.e., E_{zz}/G_{zx}.

For example, for beam (1), for $h/\Lambda = 0.4$ and $\bar{\alpha}_3 = 2.6$ (isotropic) the percentage influence of both longitudinal inertia

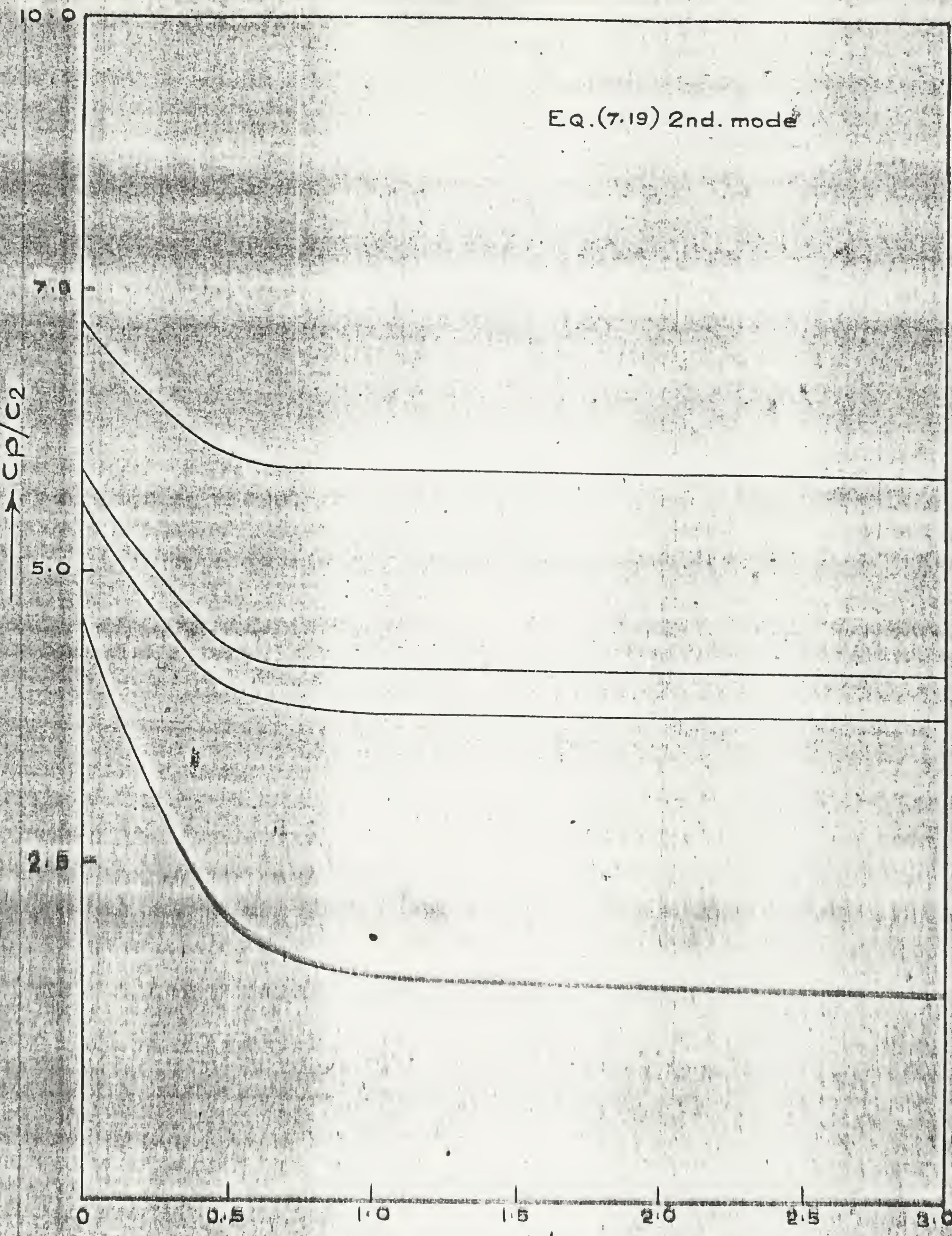

Fig. 7.6. Phase velocities for torsional waves in I-beams.
[b/h = 0.75; tf/h = 0.050; tw/h = 0.040]

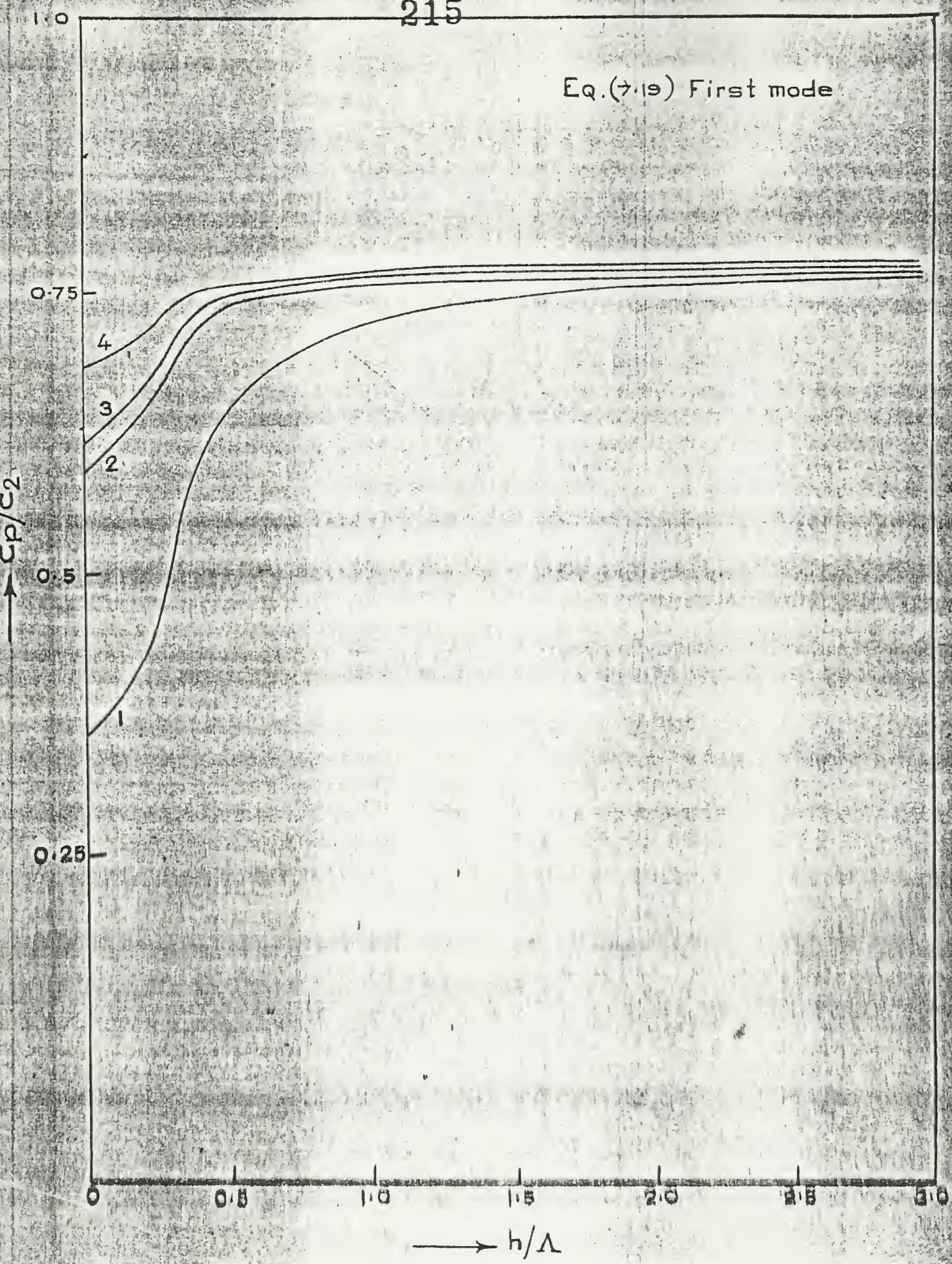

Fig. 7.7. Phase velocities for torsional waves in I-beams.
[$b/h = 1.00$; $t_f/h = 0.10$; $t_w/h = 0.050$]

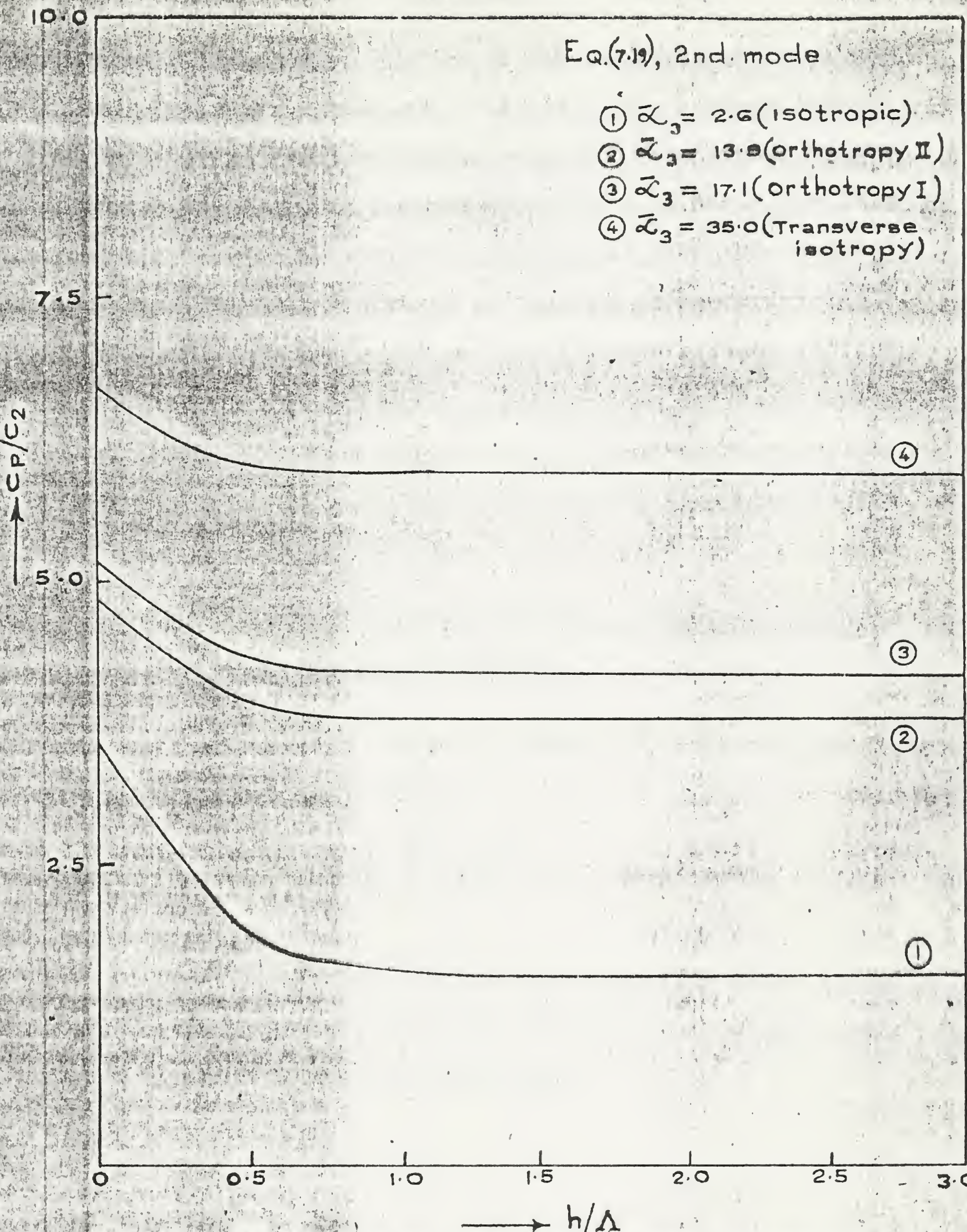

Fig. 7.8. Phase Velocities for torsional waves in I beams.
(b_f/h = 1.00, t_f/h = 0.010, t_w/h = 0.050)

and shear deformation is, $\delta_{1s} \approx 18$ percent and, that of longitudinal inertia alone is, $\delta_1 \approx 4$ percent. But these values change drastically for anisotropic member and, for instance, for $h/\Lambda = 0.4$ and $\bar{\alpha}_3 = 35.0$ (transverse isotropy), the percentage influence of both longitudinal inertia and shear deformation for the first mode, is as high as $\delta_{1s} \approx 61$ percent and that of longitudinal inertia alone is $\delta_1 \approx 4.7$ percent. Hence, it can be concluded that for some anisotropic materials, the corrections due to longitudinal inertia and shear deformation may be of one order of magnitude greater than the corrections in the isotropic case.

elastic foundation including the effects of longitudinal inertia and shear deformation. The coupled differential equations in angle of twist and warping angle governing the motion of the short thin-walled beam in torsion are derived utilizing Hamilton's principle. New frequency and normal mode equations which include the effects of time-invarient axial compressive load and elastic foundation are derived for various simple end conditions. The effects of axial load and elastic foundation, in combination with the second order influences, on the torsional frequencies and buckling loads are discussed for the case of a simply supported beam.

8.2. DERIVATION OF COUPLED EQUATIONS OF MOTION INCLUDING AXIAL LOAD AND ELASTIC FOUNDATION:

The strain energy U_4 ~~due to~~ in the Winkler-type elastic foundation is given by:

$$U_4 = \frac{1}{2} \int_0^L K_t (\phi)^2 \, dz \tag{8.1}$$

Utilizing Eqs.(4.12) and (8.1), the total strain energy U at any instant t, including the effect of Winkler-type elastic foundation can be written as:

$$U = U_1 + U_2 + U_3 + U_4$$

$$= \frac{1}{2} \int_0^L \left[GC_s \left(\frac{\partial \phi}{\partial z}\right)^2 + 2\, EI_f \left(\frac{\partial \psi}{\partial z}\right)^2 + 2\, K' A_f G \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi \right)^2 + K_t (\phi)^2 \right] dz \tag{8.2}$$

The potential energy, W, due to the time-invariant axial compressive load P is given by:

$$W = \frac{1}{2} \int_0^L \frac{PI_p}{A} \left(\frac{\partial \phi}{\partial z} \right)^2 dz \tag{8.3}$$

The total kinetic energy at time t is

$$T_k = \frac{1}{2} \int_0^L \left[\rho I_p \left(\frac{\partial \phi}{\partial t}\right)^2 + 2 \rho I_f \left(\frac{\partial \psi}{\partial t}\right)^2 \right] dz \tag{8.4}$$

which is same as Eq.(4.13).

If T_k, U and W from Eqs.(8.4), (8.2) and (8.3) are substituted into Eq.(2.1), and variations taken, and after integrating the first two terms by parts with respect to t and next five terms with respect to z, we obtain:

$$\int_{t_0}^{t_1} \int_0^L \left[\left\{ (GC_s - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} + K' A_f G h \left(\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z}\right) - K_t \phi - \rho I_p \frac{\partial^2 \phi}{\partial z} \right\} \bar{\delta} \phi + \left\{ 2 EI_f \frac{\partial^2}{\partial z^2} - 2 \rho I_f \frac{\partial^2 \psi}{\partial t^2} + 2 K' A_f G \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi\right) \right\} \bar{\delta} \psi \right] dz\, dt$$

$$+ \int_0^L \left(\rho I_p \frac{\partial \phi}{\partial t} \delta \phi + 2 \rho I_f \frac{\partial \psi}{\partial t} \bar{\delta} \psi \right) \Bigg|_{t_0}^{t_1} dz$$

$$- \int_{t_0}^{t_1} \left[\left\{ (GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} + K' A_f G h \left(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi\right) \right\} \bar{\delta} \phi + 2 EI_f \frac{\partial \psi}{\partial z} \bar{\delta} \psi \Bigg|_0^L dt = 0 \right. \tag{8.5}$$

Assuming that the values of ϕ and ψ are given at the two fixed instants, the second integral vanishes. If the boundary conditions are such that the third integral also vanishes, then we obtain the two coupled equations of motion as:

$$(GC_s - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} + K' A_f Gh(\frac{h}{2} \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \psi}{\partial z}) - K_t \phi - \rho I_p \frac{\partial^2 \phi}{\partial z^2} = 0$$

and (8.6)

$$EI_f \frac{\partial^2 \psi}{\partial z^2} + K' A_f G(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi) - \rho I_f \frac{\partial^2 \psi}{\partial t^2} = 0 \qquad (8.7)$$

8.3. NATURAL BOUNDARY CONDITIONS:

In deriving the coupled equations (8.6) and (8.7) from (8.5) it was assumed that the expression

$$\left[(GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} + K' A_f Gh(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi)\right] \bar{\delta}\phi + 2\, EI_f \frac{\partial \psi}{\partial z} \bar{\delta}\psi$$

vanishes at the ends z=0 and z=L. This condition is satisfied if at the two ends,

$$\left[(GC_s - \frac{PI_p}{A}) \frac{\partial \phi}{\partial z} + K' A_f Gh(\frac{h}{2} \frac{\partial \phi}{\partial z} - \psi)\right] \bar{\delta}\phi = 0 \qquad (8.8)$$

and

$$\frac{\partial \psi}{\partial z} \bar{\delta}\psi = 0 \qquad (8.9)$$

Eqs.(8.8) and (8.9) give the natural boundary conditions for the finite bar. Except for the case of a free end, the boundary conditions for simply supported and fixed ends remain the same as those given by Eqs.(4.19) and (4.20).

For the case of a ''free end'', the natural boundary conditions for the present problem become:

$$\frac{\partial \psi}{\partial z} = 0, \text{ and } \left(GC_s - \frac{PI_p}{A}\right)\frac{\partial \phi}{\partial z} + K' A_f G h \left(\frac{h}{2}\frac{\partial \phi}{\partial z} - \psi\right) = 0 \qquad (8.10)$$

It can be observed that the difference between Eqs.(8.10) and (4.21) for the case of the free end is due to the presence of the axial compressive load, P, acting at the shear center (or centroid) of the beam.

8.4.1. SINGLE EQUATION IN ANGLE OF TWIST:

Eliminating ψ between the coupled Equations (8.6) and (8.7), a single equation of motion in angle of twist ϕ may be obtained as:

$$\left|\frac{EI_f C_s}{K' A_f} + EC_w - \frac{PI_p EI_f}{K' A_f GA}\right| \frac{\partial^4 \phi}{\partial z^4}$$

$$- \left|\frac{E\rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2} - \frac{PI_p \rho I_f}{K' A_f GA}\right| \frac{\partial^4 \phi}{\partial z^2 \partial t^2}$$

$$- \left(GC_s + \frac{EI_f K_t}{K' A_f G} - \frac{PI_p}{A}\right)\frac{\partial^2 \phi}{\partial z^2} + \left(\rho I_p + \frac{\rho I_f K_t}{K' A_f G}\right)\frac{\partial^2 \phi}{\partial t^2}$$

$$+ \frac{\rho I_p \rho I_f}{K' A_f G}\frac{\partial^4 \phi}{\partial t^4} + K_t \phi = 0 \qquad (8.11)$$

Eq.(8.11) is the linear partial differential equation of fourth order governing the torsional vibrations and stability of a thin-walled beam resting on continuous elastic foundation.

8.4.1. ANALYSIS OF VARIOUS TERMS:

(i) Letting $C_w = \rho I_f = 0$ and $K' = \infty$, Eq.(8.11) reduces to:

$$(GC_s - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} - \rho I_p \frac{\partial^2 \phi}{\partial t^2} - K_t \phi = 0 \qquad (8.12)$$

Eq.(8.12) represents the governing differential equation of motion for the torsional vibrations and stability of a beam resting on continuous elastic foundation, based on Saint Venant torsion theory and does not include the effects of warping, longitudinal inertia and shear deformation.

(ii) If $C_w = 0$ and $K' \to \infty$, then Eq.(8.11) becomes:

$$(GC_s - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} + \frac{\rho I_f h^2}{2} \frac{\partial^4 \phi}{\partial z^2 \partial t^2} - \rho I_p \frac{\partial^2 \phi}{\partial t^2} - K_t \phi = 0 \qquad (8.13)$$

Eq.(8.13) represents the equation of motion based on Love's torsion theory and includes the effect of longitudinal inertia.

(iii) If $\rho I_f = 0$ and $K' \to \infty$, Eq.(8.11) reduces to:

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - (GC_s - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} + K_t \phi + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (8.14)$$

Eq.(8.14) is the governing differential equation of motion based on Timoshenko torsion theory which includes the effect of warping and neglects longitudinal inertia and shear deformation. It must be recalled that this equation is same as

Eq.(2.6) which is completely solved in Chapter II for various end conditions of the beam.

(iv) If $K' \to \infty$, Eq.(8.11) becomes:

$$EC_w \frac{\partial^4 \phi}{\partial z^4} - \frac{\rho I_f h^2}{2} \frac{\partial^4 \phi}{\partial z^2 \partial t^2} - (GC_s - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} + K_t \phi + \rho I_p \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{8.15}$$

Eq.(8.15) represents the governing differential equation of motion including the effects of warping and longitudinal inertia but neglecting the effect of shear deformation.

(v) If $\rho I_f = 0$, Eq.(8.11) reduces to:

$$\left[\frac{EI_f C_s}{K' A_f} + EC_w - \frac{PI_p EI_f}{K' A_f G A}\right] \frac{\partial^4 \phi}{\partial z^4} - \frac{E \rho I_p I_f}{K' A_f G} \frac{\partial^4 \phi}{\partial z^2 \partial t^2}$$

$$- (GC_s + \frac{EI_f K_t}{K' A_f G} - \frac{PI_p}{A}) \frac{\partial^2 \phi}{\partial z^2} + \rho I_p \frac{\partial^2 \phi}{\partial t^2} + K_t \phi = 0 \tag{8.16}$$

Eq.(8.16) is the equation of motion including the effects of warping and shear deformation but neglecting the effect of longitudinal inertia.

8.5. NON-DIMENSIONALIZATION AND GENERAL SOLUTION:

Eliminating ϕ in Eqs.(8.6) and (8.7) we obtain the complete differential equation in warping angle ψ as:

$$\left[\frac{EI_f C_s}{K' A_f} + EC_w - \frac{PI_p EI_f}{K' A_f GA}\right] \frac{\partial^4 \psi}{\partial z^4}$$

$$- \left[\frac{E \rho I_p I_f}{K' A_f G} + \frac{C_s \rho I_f}{K' A_f} + \frac{\rho I_f h^2}{2} - \frac{PI_p \rho I_f}{K' A_f GA}\right] \frac{\partial^4 \psi}{\partial z^2 \partial t^2}$$

$$- (GC_s + \frac{EI_f K_t}{K' A_f G} - \frac{PI_p}{A}) \frac{\partial^2 \psi}{\partial z^2} + (\rho I_p + \frac{\rho I_f K_t}{K' A_f G}) \frac{\partial^2 \psi}{\partial t^2}$$

$$+ \frac{\rho I_p \rho I_f}{K' A_f G} \frac{\partial^4 \psi}{\partial t^4} + K_t \psi = 0 \qquad (8.17)$$

Substituting Eqs.(4.30) to (4.32) and omitting the factor e^{ipt}, Eqs.(8.6), (8.7), (8.11) and (8.17) are reduced to:

$$\left[s^2(K^2 - \Delta^2) + 1\right] \bar{\phi}'' + s^2(\lambda^2 - 4\gamma^2)\bar{\phi} - (2L/h)\bar{\psi}' = 0 \qquad (8.18)$$

$$s^2 \bar{\psi}'' - (1 - \lambda^2 s^2 d^2)\bar{\psi} + (h/2L)\bar{\phi}' = 0 \qquad (8.19)$$

$$[s^2(K^2 - \Delta^2) + 1]\bar{\phi}^{iv} + [\lambda^2 a^2 d^2 + \Delta^2(1 - \lambda^2 s^2 d^2) + s^2(\lambda^2 - 4\gamma^2)]\bar{\phi}''$$

$$- (\lambda^2 - 4\gamma^2)(1 - \lambda^2 s^2 d^2)\bar{\phi} = 0 \qquad (8.20)$$

$$[s^2(K^2 - \Delta^2) + 1]\bar{\psi}^{iv} + [\lambda^2 a^2 d^2 + \Delta^2(1 - \lambda^2 s^2 d^2) + s^2(\lambda^2 - 4\gamma^2)]\bar{\psi}''$$

$$- (\lambda^2 - 4\gamma^2)(1 - \lambda^2 s^2 d^2)\bar{\psi} = 0 \qquad (8.21)$$

where primes denote differentiation with respect to Z.

The general solutions of Eqs.(8.20) and (8.21) can be found as:

$$\bar{\phi} = B_1 \cosh \alpha_3 Z + B_2 \sinh \alpha_3 Z + B_3 \cos \beta_3 Z + B_4 \sin \beta_3 Z \tag{8.22}$$

$$\bar{\psi} = B_1' \sinh \alpha_3 Z + B_2' \cosh \alpha_3 Z + B_3' \sin \beta_3 Z + B_4' \cos \beta_3 Z \tag{8.23}$$

where

$$\begin{matrix}\alpha_3 \\ \beta_3\end{matrix} = \frac{1}{\sqrt{2}\left[s^2(K^2-\Delta^2)+1\right]^{1/2}}\left\{\mp\left[\lambda^2a^2d^2+\Delta^2(1-\lambda^2s^2d^2)+s^2(\lambda^2-4\gamma^2)\right]\right.$$
$$\left. + \left[\left[\lambda^2a^2d^2+\Delta^2(1-\lambda^2s^2d^2)-s^2(\lambda^2-4\gamma^2)\right]^2+4(\lambda^2-4\gamma^2)\right]^{1/2}\right\}^{1/2} \tag{8.24}$$

and

$$\left\{\left[\lambda^2a^2d^2+\Delta^2(1-\lambda^2s^2s^2)-s^2(\lambda^2-4\gamma^2)\right]^2+4(\lambda^2-4\gamma^2)\right\}^{1/2}$$
$$> \left[\lambda^2a^2d^2+\Delta^2(1-\lambda^2s^2d^2)+s^2(\lambda^2-4\gamma^2)\right]$$

is assumed.

In case

$$\left\{\left[\lambda^2a^2d^2+\Delta^2(1-\lambda^2s^2d^2)-s^2(\lambda^2-4\gamma^2)\right]^2+4(\lambda^2-4\gamma^2)\right\}^{1/2}$$
$$< \left[\lambda^2a^2d^2+\Delta^2(1-\lambda^2s^2d^2)+s^2(\lambda^2-4\gamma^2)\right]$$

we write

$$\alpha_3 = \frac{1}{\sqrt{2}\left[s^2(K^2-\Delta^2)+1\right]^{1/2}}\left\{\left[\lambda^2a^2d^2+\Delta^2(1-\lambda^2s^2d^2)+s^2(\lambda^2-4\gamma^2)\right]\right.$$
$$\left. - \left[\left[\lambda^2a^2d^2+\Delta^2(1-\lambda^2s^2d^2)-s^2(\lambda^2-4\gamma^2)\right]^2+4(\lambda^2-4\gamma^2)\right]^{1/2}\right\}^{1/2}$$
$$= i\,\alpha_3' \tag{8.25}$$

Then Eqs.(8.22) and (8.23) are replaced by

$$\bar{\phi} = B_1 \cos \alpha_3' Z + iB_2 \sin \alpha_3' Z + B_3 \cos \beta_3 Z + B_4 \sin \beta_3 Z \qquad (8.26)$$

$$\bar{\psi} = iB_1' \sin \alpha_3' Z + B_2' \cos \alpha_3' Z + B_3' \sin \beta_3 Z + B_4' \cos \beta_3 Z \qquad (8.27)$$

Solutions of Eqs.(8.22) and (8.23) or (8.26) and (8.27) are naturally the solutions of the original coupled equations (8.6) and (8.7).

Only one half of the constants in Eqs.(8.22) and (8.23) are independent. They are related by Eqs.(8.6) and (8.7) as follows:

$$B_1 = \frac{2L}{h\alpha_3} \left[1 - s^2 (\alpha_3^2 + \lambda^2 d^2) \right] B_1' \qquad (8.28)$$

$$B_2 = \frac{2L}{h\alpha_3} \left[1 - s^2 (\alpha_3^2 + \lambda^2 d^2) \right] B_2' \qquad (8.29)$$

$$B_3 = -\frac{2L}{h\beta_3} \left[1 + s^2 (\beta_3^2 - \lambda^2 d^2) \right] B_3' \qquad (8.30)$$

$$B_4 = \frac{2L}{h\beta_3} \left[1 + s^2 (\beta_3^2 - \lambda^2 d^2) \right] B_4' \qquad (8.31)$$

or

$$B_1' = \frac{h}{2L\alpha_3} \left\{ \alpha_3^2 \left[s^2 (K^2 - \Delta^2) + 1 \right] + s^2 (\lambda^2 - 4\gamma^2) \right\} B_1 \qquad (8.32)$$

$$B_2' = \frac{h}{2L\,\alpha_3} \left\{ \alpha_3^2 \left[s^2 (K^2 - \Delta^2) + 1 \right] + s^2 (\lambda^2 - 4\gamma^2) \right\} B_2 \qquad (8.33)$$

$$B_3' = -\frac{h}{2L\,\beta_3} \left\{ \beta_3^2 \left[s^2 (K^2 - \Delta^2) + 1 \right] - s^2 (\lambda^2 - 4\gamma^2) \right\} B_3 \qquad (8.34)$$

$$B_4' = \frac{h}{2L\,\beta_3} \quad \beta_3^2 \left[s^2(K^2-\triangle^2)+1 \right] - s^2(\lambda^2-4\gamma^2) \quad B_4 \qquad (8.35)$$

8.6. FREQUENCY OR BUCKLING LOAD EQUATIONS AND MODAL FUNCTIONS:

In section 8.3, natural boundary conditions for the present problem are discussed. By combining these conditions in pairs, many types of single-span beams can be analyzed. In terms of non-dimensional parameters, the boundary conditions for a ''free end'' can be written as:

$$\bar{\psi}' = 0, \quad \left[s^2(K^2-\triangle^2) + 1 \right] \bar{\phi}' - (2L/h)\bar{\psi} = 0 \qquad (8.36)$$

The application of appropriate boundary conditions (4.56), (4.57) and (8.36) and, relations of integration constants (8.28) to (8.35) to Eqs.(8.22) and (8.23) yields for each type of beam a set of four constants B_1 to B_4 with or without primes. In order that solutions other than zero may exist the determinant of the coefficients of B's must be equal to zero. This leads to the frequency equations in each case and the roots of these frequency or buckling load equations, λ_i, $i = 1,2,3,\ldots n$, or $\triangle^2_{cr}$, give the eigen values of the problem. The corresponding modal functions, $\bar{\phi}_i$ and $\bar{\psi}_i$ can be obtained accordingly.

8.6.1. SIMPLY SUPPORTED BEAM:

The boundary conditions for a beam simply supported at both ends are:

$$\bar{\phi} = \bar{\psi}' = 0 \qquad \text{at} \quad Z = 0$$

and

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at} \quad Z = 1$$

For the boundary conditions at Z = 0, Eqs.(8.22) and (8.23) give:

$$B_1 + B_3 = 0 \tag{8.37}$$

$$\left\{\alpha_3^2 \left[s^2(K^2-\Delta^2) + 1\right] + s^2(\lambda^2 - 4\gamma^2)\right\} B_1$$

$$- \left\{\beta_3^2 \left[s^2(K^2-\Delta^2)+ 1\right] - s^2(\lambda^2 - 4\gamma^2)\right\} B_3 = 0 \tag{8.38}$$

Since the secular determinant, ie.,

$$\left[s^2(K^2-\Delta^2) + 1 \right] (\alpha_3^2 + \beta_3^2) \neq 0,$$

therefore it follows that $B_1 = B_3 = 0$. (8.39)

For the second pair of conditions at Z = 1, Eqs.(8.22) and (8.23) give:

$$B_2 \sinh \alpha_3 + B_4 \sin \beta_3 = 0 \tag{8.40}$$

and

$$\left\{ \alpha_3^2 \left[s^2(K^2-\Delta^2)+ 1\right]+ s^2(\lambda^2 - 4\gamma^2)\right\} B_2 \sinh \alpha_3$$

$$- \beta_3^2 \left[s^2(K^2-\Delta^2)+ 1\right]- s^2(\lambda^2 - 4\gamma^2)\Big\} B_4 \sin \beta_3 = 0 \tag{8.41}$$

For a non-trivial solution, the secular determinant must vanish. This gives the characterestic equation:

$$\left[s^2(K^2-\Delta^2)+ 1\right] (\alpha_3^2 + \beta_3^2) \sinh \alpha_3 \sin \beta_3 = 0 \tag{8.42}$$

Since $\left[s^2(K^2-\Delta^2)+1\right](\alpha_3^2+\beta_3^2) \neq 0$

and

$$\alpha_3 \neq 0,$$

From Eq.(8.42) we have

$$\beta_3 = n\pi, \; n = 1,2,3,\ldots. \qquad (8.43)$$

which leads to the main solution of the problem.

Letting $\beta_3^2 = n^2\pi^2$ in Eq.(8.24), the frequency equation in λ^2 is obtained as:

$$s^2d^2\lambda^4 - \lambda^2 \left\{1 + n^2\pi^2\left[s^2+d^2+s^2d^2(K^2-\Delta^2)\right] + 4s^2d^2\gamma^2\right\} + \left\{n^4\pi^4\left[s^2(K^2-\Delta^2)+1\right] + n^2\pi^2(K^2-\Delta^2) + 4\gamma^2(1+n^2\pi^2s^2)\right\} = 0 \qquad (8.44)$$

This equation gives two real positive roots:

$$\lambda^2_{mn} = \frac{1}{2s^2d^2}\left[\left[1+n^2\pi^2\left\{s^2+d^2+s^2d^2(K^2-\Delta^2)\right\} + 4s^2d^2\gamma^2\right] + (-1)^m\left\{\left[1+n^2\pi^2\left\{s^2-d^2-s^2d^2(K^2-\Delta^2)\right\} - 4s^2d^2\gamma^2\right]^2 + 4n^2\pi^2d^2\right\}^{\frac{1}{2}}\right] \qquad (8.45)$$

This frequency equation (8.45) in λ^2, has an infinite number of roots which in general represent two coupled frequency spectra.

Using Eqs.(8.43), (8.40) and (8.41), one gets:

$$B_2 = 0 \qquad (8.46)$$

The modal functions are obtained from Eqs.(4.22) and (4.23) with B's given by Eqs.(8.39) and (8.46). These are given as:

$$\bar{\phi}_{mn} = \sin n\pi Z \qquad (8.47)$$

$$\bar{\psi}_{mn} = \frac{h}{2n\pi L}\left\{n^2\pi^2\left[s^2(K^2-\Delta^2)+1\right]-s^2(\lambda^2_{mn}-4\gamma^2)\right\}\cos n\pi Z \qquad (8.48)$$

where λ^2_{mn} being given by (8.45).

The second spectrum appears at higher frequencies, greater than the critical frequency λ_c given by

$$\lambda_c^2 = 1/s^2d^2$$

and is due to interaction between shear deformation and longitudinal inertia. It should be mentioned here that for the range of values of the dimensionless parameters covered in this chapter, λ is less than λ_c.

For the case, $\lambda > \lambda_c$, it is convenient to use $\alpha_3 = i\alpha_3'$ and, the characterestic frequency equation (8.42) transforms to:

$$\sin \alpha_3' \ \sin \beta_3 = 0 \qquad (8.49)$$

Hence, in case there is any extension from there on for λ beyond λ_c ie., $\lambda^2s^2d^2>1$, care should be taken to account for the frequencies of the second spectrum which can be obtained from Eq.(8.49).

By putting $s^2= d^2= 0$, in Eq.(8.44), the equation for the the frequency parameter λ, neglecting the effects of shear deformation and longitudinal inertia, can be obtained as:

$$\lambda^2 = n^2\pi^2(n^2\pi^2 + K^2 - \Delta^2) + 4\gamma^2 \qquad (8.50)$$

which is the same as Eq.(2.47) derived in Chapter-II utilizing Timoshenko torsion theory.

8.6.2. FIXED-FIXED BEAM:

For a beam clamped at both ends, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 0$$

and

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 1.$$

Applying the above boundary conditions to the general solutions, Eqs.(8.22) and (8.23), the frequency equation, for the first set ($\lambda < \lambda_c$) can be obtained as:

$$2-2 \cosh \alpha_3 \cos \beta_3 + \frac{(1-\delta_1^2\theta_1^2)}{\delta_1\theta_1} \sinh \alpha_3 \sin \beta_3 = 0 \qquad (8.51)$$

where

$$\delta_1 = \alpha_3/\beta_3 \qquad (8.52)$$

and

$$\theta_1 = \frac{\beta_3^2\left|s^2(K^2-\Lambda^2)+1\right| - s^2(\lambda^2-4\gamma^2)}{\alpha_3^2\left|s^2(K^2-\Lambda^2)+1\right| + s^2(\lambda^2-4\gamma^2)} \qquad (8.53)$$

The frequency equation for the second set ($\lambda > \lambda_c$) is:

$$2-2\cos\alpha_3'\cos\beta_3+\frac{(1+\delta_2^2\theta_2^2)}{\delta_2\theta_2}\sin\alpha_3'\sin\beta_3 = 0 \tag{8.54}$$

where

$$\delta_2 = \alpha_3'/\beta_3 \tag{8.55}$$

and

$$\theta_2 = -\frac{\beta_3^2\left[s^2(K^2-\triangle^2)+1\right]-s^2(\lambda^2-4\gamma^2)}{\alpha_3^2\left|s^2(K^2-\triangle^2)+1\right|-s^2(\lambda^2-4\gamma^2)} \tag{8.56}$$

The modal functions for the first set are given by:

$$\bar{\phi} = D(\cosh\alpha_3 Z+\delta_1\eta_1^*\theta_1\sinh\alpha_3 Z-\cos\beta_3 Z+\eta_1^*\sin\beta_3 Z) \tag{8.57}$$

$$\bar{\psi} = H(\cosh\alpha_3 Z+\frac{\mu_1^*}{\delta_1\theta_1}\sinh\alpha_3 Z-\cos\beta_3 Z+\mu_1^*\sin\beta_3 Z) \tag{8.58}$$

where

$$\eta_1^* = \frac{-\cosh\alpha_3+\cos\beta_3}{\delta_1\theta_1\sinh\alpha_3-\sin\beta_3} \tag{8.59}$$

$$\mu_1^* = \frac{-\cosh\alpha_3+\cos\beta_3}{(1/\delta_1\theta_1)\sinh\alpha_3+\sin\beta_3} \tag{8.60}$$

The modal functions for the second set are:

$$\bar{\phi} = D(\cos\alpha_3' Z-\delta_2\eta_2^*\theta_2\sin\alpha_3' Z-\cos\beta_3 Z+\eta_2^*\sin\beta_3 Z) \tag{8.61}$$

$$\bar{\psi} = H(\cos\alpha_3' Z+\frac{\mu_2^*}{\delta_2\theta_2}\sin\alpha_3' Z-\cos\beta_3 Z+\mu_2^*\sin\beta_3 Z) \tag{8.62}$$

where

$$\eta_2^* = \frac{\cos\alpha_3'-\cos\beta_3}{\delta_2\theta_2\sin\alpha_3'+\sin\beta_3} \tag{8.63}$$

$$\mu_2^* = \frac{-\cosh \alpha_3' + \cos \beta_3}{(1/\delta_2\theta_2)\sin \alpha_3' + \sin \beta_3} \qquad (8.64)$$

Since the coefficients in $\bar{\phi}$ and $\bar{\psi}$ of Eqs.(8.22) and (8.23) are related, the coefficients D and H, that appear in the modalfunctions given above, are connected through any one of the Eqs.(8.28) to (8.31) or (8.32) to (8.35).

8.6.3. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

With the end Z = 0, taken as clamped end, and with the end Z = 1 as the simply supported end, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 0$$

and

$$\bar{\phi} = \bar{\psi} = 0 \quad \text{at} \quad Z = 1$$

The frequency equation obtained from applying the above boundary condition to the general solutions, Eqs.(8.22) and (8.23) for the first set ($\lambda < \lambda_c$) is given by:

$$\delta_1\theta_1 \tanh \alpha_3 - \tan \beta_3 = 0 \qquad (8.65)$$

The frequency equation for the second set ($\lambda > \lambda_c$) is:

$$\delta_2\theta_2 \tan \alpha_3' + \tan \beta_3 = 0 \qquad (8.66)$$

The modal functions for the first set are given by:

$$\bar{\phi} = D(\cosh \alpha_3 Z - \coth \alpha_3 \sinh \alpha_3 Z - \cos \beta_3 Z + \cot \beta_3 \sin \beta_3 Z) \qquad (8.67)$$

$$\bar{\psi} = H(\cosh \alpha_3 Z + \frac{\mu_3^*}{\delta_1\theta_1} \sinh \alpha_3 Z - \cos \beta_3 Z + \mu_3^* \sin \beta_3 Z) \qquad (8.68)$$

where

$$\mu_3^* = \frac{-(\delta_1 \sinh \alpha_3 + \sin \beta_3)}{(1/\theta_1)\cosh \alpha_3 + \cos \beta_3} \tag{8.69}$$

The modal functions for the second set are:

$$\bar{\phi} = D(\cos \alpha_3' Z - \cot \alpha_3' \sin \alpha_3' Z - \cos \beta_3 Z + \cot \beta_3 \sin \beta_3 Z) \tag{8.70}$$

$$\bar{\psi} = H(\cos \alpha_3' Z - \frac{\eta_3^*}{\delta_2 \theta_2} \sin \alpha_3' Z - \cos \beta_3 Z + \eta_3^* \sin \beta_3 Z) \tag{8.71}$$

where

$$\eta_3^* = \frac{\delta_2 \sin \alpha_3' - \sin \beta_3}{(1/\theta_2)\cos \alpha_3' + \cos \beta_3} \tag{8.72}$$

8.6.4. CANTILEVER BEAM WITH ONE END FIXED AND FREE AT THE OTHER:

For a cantilever beam built in rigidly at the end $Z = 0$ so that warping is completely prevented, and with a free end at $Z = 1$, the boundary conditions are:

$$\bar{\phi} = \bar{\psi} = 0 \qquad \text{at} \quad Z = 0$$

and

$$\bar{\psi}' = 0, \ \left[s^2(K^2 - \Delta^2) + 1 \right] \bar{\phi}' - (2L/h)\bar{\psi} = 0 \text{ at } Z = 1.$$

The frequency equation for the first set, in this case, can be obtained as:

$$2 + \frac{(1+\theta_1^2)}{\theta_1} \cosh \alpha_3 \cos \beta_3 - \frac{(1-\delta_1^2)}{\delta_1} \sinh \alpha_3 \sin \beta_3 = 0 \tag{8.73}$$

The frequency equation for the second set is given by:

$$2 + \frac{(1+\theta_2^2)}{\theta_2} \cos \alpha_3' \cos \beta_3 - \frac{(1+\delta_2^2)}{\delta_2} \sin \alpha_3' \sin \beta_3 = 0 \qquad (8.74)$$

The modal functions for the first set are:

$$\bar{\phi} = D(\text{Cosh } \alpha_3 Z - \delta_1 \theta_1 \eta_4^* \sinh \alpha_3 Z - \cos \beta_3 Z + \eta_4^* \sin \beta_3 Z) \qquad (8.75)$$

$$\bar{\psi} = H(\cosh \alpha_3 Z + \frac{\mu_4^*}{\delta_1 \theta_1} \sin h\, \alpha_3 Z - \cos \beta_3 Z + \mu_4^* \sin \beta_3 Z) \qquad (8.76)$$

where

$$\eta_4^* = \frac{(1/\delta_1) \sinh \alpha_3 - \sin \beta_3}{\theta_1 \cosh \alpha_3 + \cos \beta_3} \qquad (8.77)$$

$$\mu_4^* = -\frac{(\delta_1 \sinh \alpha_3 + \sin \beta_3)}{(1/\theta_1) \cosh \alpha_3 + \cos \beta_3} \qquad (8.78)$$

The modal functions for the second set are:

$$\bar{\phi} = D(\cos \alpha_3' Z + \delta_2 \theta_2 \eta_5^* \sin \alpha_3' Z - \cos \beta_3 Z + \eta_5^* \sin \beta_3 Z) \qquad (8.79)$$

$$\bar{\psi} = H(\cos \alpha_3' Z - \frac{\mu_5^*}{\delta_2 \theta_2} \sin \alpha_3' Z - \cos \beta_3 Z + \mu_5^* \sin \beta_3 Z) \qquad (8.80)$$

where

$$\eta_5^* = \frac{(1/\delta_2) \sin \alpha_3' - \sin \beta_3}{\theta_2 \cos \alpha_3' + \cos \beta_3} \qquad (8.81)$$

$$\mu_5^* = \frac{\delta_2 \sin \alpha_3' - \sin \beta_3}{(1/\theta_2) \cos \alpha_3' + \cos \beta_3} \qquad (8.82)$$

8.6.5. CANTILEVER BEAM WITH ONE END SIMPLY SUPPORTED AND FREE AT THE OTHER:

For a cantilever beam simply supported at the end Z = 0 and free at Z = 1, the boundary conditions are:

$$\bar{\phi} = \bar{\psi}' = 0 \quad \text{at } Z = 0,$$

and

$$\bar{\psi}' = 0, \; \left[s^2(K^2 - \Delta^2) + 1 \right] \bar{\phi}' - (2L/h)\bar{\psi} = 0 \text{ at } Z = 1.$$

The frequency equation for the first set, in this case becomes:

$$\delta_1 \tanh \alpha_3' - \theta_1 \tan \beta_3 = 0 \qquad (8.83)$$

The frequency equation for the second set is given by:

$$\delta_2 \tan \alpha_3' + \theta_2 \tan \beta_3 = 0 \qquad (8.84)$$

The modal functions for the first set are:

$$\bar{\phi} = \frac{\delta_1 \cos \beta_3}{\cosh \alpha_3} \sinh \alpha_3 Z + \sin \beta_3 Z \qquad (8.85)$$

$$\bar{\psi} = \frac{\sin \beta_3}{\delta_1 \sinh \alpha_3} \cosh \alpha_3 Z + \cos \beta_3 Z \qquad (8.86)$$

The modal functions for the second set can be obtained as:

$$\bar{\phi} = -\frac{\delta_2 \cos \beta_3}{\cos \alpha_3'} \sin \alpha_3' Z + \sin \beta_3 Z \qquad (8.87)$$

$$\bar{\psi} = -\frac{\sin \beta_3}{\delta_2 \sin \alpha_3'} \cos \alpha_3' Z + \cos \beta_3 Z \qquad (8.88)$$

8.6.6. BEAM WITH FREE ENDS:

In the case of a beam which is free at both ends, the boundary conditions are:

$$\bar{\psi}' = 0, \quad \left[s^2(K^2-\Delta^2)+1 \right] \bar{\phi}' - (2L/h)\bar{\psi} = 0 \text{ at } Z = 0,$$

and

$$\bar{\psi}' = 0, \quad \left[s^2(K^2-\Delta^2)+1 \right] \bar{\phi}' - (2L/h)\bar{\psi} = 0 \text{ at } Z = 1.$$

The frequency equation for the first set, in this case can be obtained as:

$$2-2 \cosh \alpha_3 \cos \beta_3 + \frac{(\theta_1^2-\delta_1^2)}{\delta_1 \theta_1} \sinh \alpha_3 \sin \beta_3 = 0 \qquad (8.89)$$

The frequency equation for the second set is given by:

$$2-2 \cos \alpha_3' \cos \beta_3 + \frac{(\theta_2^2+\delta_2^2)}{\delta_2^2 \theta_2} \sin \alpha_3' \sin \beta_3 = 0 \qquad (8.90)$$

The modal functions for the first set can be obtained as:

$$\bar{\phi} = D(\cosh \alpha_3 Z + \eta_6^* \delta_1 \sinh \alpha_3 Z + (1/\theta_1)\cos \beta_3 Z + \eta_6^* \sin \beta_3 Z) \qquad (8.91)$$

$$\bar{\psi} = H(\cosh \alpha_3 Z - \frac{\eta_6^*}{\delta_1} \sinh \alpha_3 Z + \theta_1 \cos \beta_3 Z + (1/\eta_6^*)\sin \beta_3 Z) \qquad (8.92)$$

where

$$\eta_6^* = \frac{\cosh \alpha_3 - \cos \beta_3}{\delta_1 \sinh \alpha_3 - \theta_1 \sin \beta_3} \qquad (8.93)$$

The modal functions for the second set are given by:

$$\bar{\phi} = D(\cos \alpha_3' Z - \delta_2 \mu_6^* \sin \alpha_3' Z + (1/\theta_2)\cos \beta_3 Z + \mu_6^* \sin \beta_3 Z) \qquad (8.94)$$

$$\bar{\psi} = H(\cos \alpha_3' Z - (\mu_6^*/\delta_2)\sin \alpha_3' Z + \theta_2 \cos \beta_3 Z + (1/\mu_6^*) \sin \beta_3 Z) \qquad (8.95)$$

where

$$\mu_6^* = - \frac{\cos \alpha_3' - \cos \beta_3}{\delta_2 \sin \alpha_3' + \theta_2 \sin \beta_3} \qquad (8.96)$$

8.7. APPROXIMATE SOLUTIONS BY GALERKIN'S TECHNIQUE:

Except for the simply supported beam, the frequency equations for other boundary conditions derived in the section (8.6) can be observed to be highly transcendental and are solved on a digital computer only by lengthy trial-and-error method. An attempt has been made in this section to derive approximate expressions for the torsional frequencies and buckling loads of fixed-fixed beam and of a beam fixed at one end and simply supported at the other, utilizing ~~the~~ Galerkin's technique.

8.7.1. FIXED-FIXED BEAM:

To satisfy the boundary conditions in this case, the normal function of angle of twist $\bar{\phi}$ can be assumed in the form

$$\bar{\phi} = \sum_{n=1}^{\infty} B_n(1 - \cos 2n\pi Z) \qquad (8.97)$$

Substituting Eq.(8.97) in the differential Equation (8.20) and using ~~the~~ Galerkin's technique, expression for the frequency parameter λ^2, in this can be obtained as:

$$3\lambda^4 s^2 d^2 - \lambda^2\left\{3+4n^2\pi^2\left[s^2+d^2+s^2d^2(K^2-\triangle^2)\right]+12\,s^2d^2\gamma^2\right\}$$
$$+\left\{16\,n^4\pi^4\left[s^2(K^2-\triangle^2)+1\right]+4n^2\pi^2(K^2-\triangle^2)+4\gamma^2(3+4n^2\pi^2s^2)\right\}=0 \quad (8.98)$$

Eq.(8.98) gives two real positive roots given by

$$\lambda^2_{mn} = \frac{1}{3s^2d^2}\left[3+4n^2\pi^2\left[s^2+d^2+s^2d^2(K^2-\triangle^2)\right]+12s^2d^2\gamma^2\right]$$
$$+(-1)^m\left[\left\{3+4n^2\pi^2\left[s^2+d^2+s^2d^2(K^2-\triangle^2)\right]+12s^2d^2\gamma^2\right\}^2\right.$$
$$-12s^2d^2\left\{16n^4\pi^4\left|s^2(K^2-\triangle^2)+1\right|+4n^2\pi^2(K^2-\triangle^2)\right.$$
$$\left.\left.+4\gamma^2(3+4n^2\pi^2s^2)\right\}\right]^{1/2}\Bigg\} \quad (8.99)$$

For a beam not vibrating, ie., $\lambda = 0$, the expression for the buckling load can be obtained from Eq.(8.98) as

$$\triangle^2_{cr} = K^2 + \left[\frac{4\pi^4+\gamma^2(3+4\pi^2s^2)}{\pi^2(1+4\pi^2s^2)}\right] \quad (8.100)$$

If the effect of shear deformation is neglected, ie., $s^2 = 0$, Eq.(8.100) reduces to:

$$\triangle^2_{cr} = 4\,\pi^2 + K^2 + (3/\pi^2)\gamma^2 \quad (8.101)$$

which is same as Eq.(2.74) obtained by utilizing Timoshenko torsion theory.

If the effects of longitudinal inertia and shear deformation are neglected, ie., $s^2 = d^2 = 0$, Eq.(8.98) yields:

$$\lambda = 2\left[(n^2\pi^2/3)(4\,n^2\pi^2 + K^2 - \Delta^2) + \gamma^2\right]^{1/2} \tag{8.102}$$

which is same as Eq.(2.73).

8.7.2. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER:

To satisfy the boundary conditions in this case, the normal function of angle of twist $\bar{\phi}$ can be taken as:

$$\bar{\phi} = \sum_{n=1}^{\infty} D_n\left(\cos\frac{n\pi}{2}Z - \cos\frac{3n\pi}{2}Z\right) \tag{8.103}$$

Substituting Eq.(8.103) in the differential Equation (8.20) and using the Galerkin's technique, the expression for the frequency parameter λ^2, in this case can be obtained as:

$$16\lambda^4 s^2 d^2 - \lambda^2\left\{16 + 20\,n^2\pi^2\left[s^2 + d^2 + s^2 d^2(K^2 - \Delta^2)\right] + 64\,s^2 d^2\gamma^2\right\} + \left\{41\,n^4\pi^4\left[s^2(K^2 - \Delta^2) + 1\right] + 20\,n^2\pi^2(K^2 - \Delta^2) + 16\,\gamma^2(4 + 5\,n^2\pi^2 s^2)\right\} = 0 \tag{8.104}$$

From Eq.(8.104) we have

$$\lambda^2_{mn} = \frac{1}{16\,s^2 d^2}\Bigg\{\left[16 + 20n^2\pi^2\left[s^2 + d^2 + s^2 d^2(K^2 - \Delta^2)\right] + 64\,s^2 d^2\gamma^2\right] + (-1)^m\Bigg[\left\{16 + 20n^2\pi^2\left[s^2 + d^2 + s^2 d^2(K^2 - \Delta^2)\right] + 64\,s^2 d^2\gamma^2\right\}^2 - 64\,s^2 d^2\left\{41n^4\pi^4\left[s^2(K^2 - \Delta^2) + 1\right] + 20\,n^2\pi^2(K^2 - \Delta^2) + 16\,\gamma^2(4 + 5\,n^2\pi^2 s^2)\right\}\Bigg]^{1/2}\Bigg\} \tag{8.105}$$

For a beam not vibrating, ie., $\lambda = 0$, and the expression for the buckling load can be obtained from Eq.(8.104) as:

$$\Delta^2_{cr} = K^2 + \left[\frac{2.05\ \pi^4 + 0.8\gamma^2(4+5\ \pi^2 s^2)}{\pi^2(1+2.05\ \pi^2 s^2)} \right] \qquad (8.106)$$

If the effect of shear deformation is neglected, ie., $s^2 = 0$, Eq.(8.106) reduces to:

$$\Delta^2_{cr} = 2.05\ \pi^2 + K^2 + (3.2/\pi^2)\gamma^2 \qquad (8.107)$$

which is same as Eq.(2.77) derived by utilizing Timoshenko torsion theory.

If the effects of longitudinal inertia and shear deformation are neglected, ie., $s^2 = d^2 = 0$, Eq.(8.104) yields:

$$\lambda = \left[1.25\ n^2\pi^2(2.05\ n^2\pi^2 + K^2 - \Delta^2) + 4\gamma^2 \right]^{1/2} \qquad (8.108)$$

which is same as Eq.(2.76).

8.8. LIMITING CONDITIONS:

The limiting conditions at which the combined influence of the axial compressive load and elastic foundation on the torsional frequency becomes zero, for some cases are as follows:

(1) Simply-Supported Beam:

From Eq.(8.44) we get two limiting conditions in this case. They are:

(a) $sd\gamma = 0.5\ n\pi\Delta$ (8.109)

(b) $\gamma = 0.5\ n\pi\Delta$ (8.110)

(2) Fixed-Fixed Beam: From Eq.(8.98) the limiting conditions in this case are:

$$\text{(a)}\ \sqrt{3}\, sd\,\nu = n\pi\Delta \tag{8.111}$$

$$\text{(b)}\ \nu = n\pi\Delta \left[\frac{1+4\,n^2\pi^2 s^2}{3+4\,n^2\pi^2 s^2}\right]^{1/2} \tag{8.112}$$

(3) Beam fixed at one end and Simply supported at the other:

From Eq.(8.104) the limiting conditions in this case are:

$$\text{(a)}\ 4\, sd\,\nu = \sqrt{5}\, n\pi\Delta \tag{8.113}$$

$$\text{(b)}\ \nu = 0.559\, n\pi\Delta \left[\frac{1+2.05\,n^2\pi^2 s^2}{1+1.25\,n^2\pi^2 s^2}\right]^{1/2} \tag{8.114}$$

If the effect of shear deformation is neglected, ie., $s^2 = 0$, Eqs.(8.112) and (8.114) reduces to Eqs.(2.79) and (2.80) derived previously.

For the above relations in various cases between ν and Δ there will be no influence of axial load and elastic foundation on the torsional frequency of vibration. This can be observed to be due to the opposite nature of their individual effects and these individual effects get nullified at these limiting conditions for various cases.

8.9. RESULTS AND CONCLUSIONS:

In this section, the results obtained on IBM 1130 Computer are presented in Tables 8.1 to 8.16 to show the effects of various non-dimensional parameters on the buckling loads and torsional frequencies of simply supported, clamped-clamped and clamped-simply supported beams resting on elastic foundation. Extensive design data are made available in these tables. The main interest is to find the influences of shear deformation and longitudinal inertia on the frequencies of vibration of a short thin-walled beam resting on continuous elastic foundation and subjected to an axial compressive load.

The values of the torsional buckling load Δ_{cr} for the three boundary conditions are given in Table 8.1 for various values of the warping parameter K and shear parameter s. It is well known that the effect of increase in the value of K is to increase the buckling load considerably. From Table 8.1, we observe that for any constant value of K, the effect of increase in the value of s is to decrease the torsional buckling load, and that this reduction becomes significant for values of $K \leq 1$. Also, the effect of shear deformation in reducing the buckling load is comparitively considerable in clamped-clamped beams than in other cases.

The results showing the combined effects of axial compressive load, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) are given in Tables 8.2, 8.6 and 8.10, for values of K = 0.01 and s = 2d. The percentage

TABLE - 8.1

Effects of shear deformation and elastic foundation on the torsional buckling loads of simply supported, clamped-clamped and clamped-simply supported thin-walled beams of open section.

γ	s	Simply supported beam			Clamped-clamped beam			Clamped-simply supported beam		
		K=0.01	K=1.00	K=10.00	K=0.01	K=1.00	K=10.00	K=0.01	K=1.00	K=10.00
0	0.04	3.117	3.274	10.474	6.094	6.175	11.710	4.427	4.539	10.936
	0.08	3.047	3.207	10.454	5.614	5.702	11.468	4.232	4.349	10.859
	0.10	2.997	3.160	10.440	5.320	5.413	11.327	4.102	4.222	10.809
4	0.04	4.025	4.147	10.780	6.466	6.542	11.908	4.972	5.072	11.168
	0.08	3.971	4.095	10.760	5.977	6.060	11.650	4.782	4.886	11.085
	0.10	3.933	4.058	10.746	5.679	5.766	11.500	4.656	4.762	11.031
8	0.04	5.971	6.054	11.647	7.471	7.538	12.483	6.332	6.411	11.836
	0.08	5.935	6.018	11.628	6.954	7.025	12.180	6.143	6.224	11.736
	0.10	5.909	5.993	11.616	6.640	6.715	12.004	6.018	6.100	11.671
12	0.04	8.251	8.311	12.964	8.898	8.954	13.385	8.107	8.168	12.873
	0.08	8.225	8.285	12.948	8.331	8.391	13.015	7.907	7.970	12.748
	0.10	8.206	8.267	12.936	7.988	8.051	12.799	7.775	7.839	12.667

T A B L E - 8.2

Effects of axial compressive load longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams (ν= 0, K=0.01 and s=2d).

Δ	s	d	Values of λ^2 and $q = \lambda/\lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0.5	0.00	0.00	94.944	1.000	1548.694	1.000	7868.009	1.000	24897.484	1.000
	0.04	0.02	92.977	0.990	1436.073	0.963	6702.987	0.923	19091.711	0.876
	0.08	0.04	87.927	0.962	1186.118	0.875	4726.499	0.775	11592.957	0.682
	0.10	0.05	84.469	0.943	1053.103	0.825	3900.204	0.704	9041 .121	0.603
1.0	0.00	0.00	87.541	1.000	1519.085	1.000	7801.389	1.000	24779.051	1.000
	0.04	0.02	85.784	0.990	1406.831	0.962	6638.099	0.922	18977.887	0.875
	0.08	0.04	80.626	0.960	1157.668	0.873	4663.736	0.773	11482.309	0.681
	0.10	0.05	77.213	0.939	1024.978	0.821	3837.396	0.701	8930.559	0.600
1.5	0.00	0.00	75.204	1.000	1469.737	1.000	7690.355	1.000	24581.656	1.000
	0.04	0.02	73.329	0.987	1358.195	0.961	6530.017	0.921	18788.234	0.874
	0.08	0.04	68.469	0.954	1110.274	0.869	4559.131	0.770	11297.918	0.678
	0.10	0.05	65.126	0.931	978.090	0.816	3734.035	0.697	8746.240	0.596
2.0	0.00	0.00	57.932	1.000	1400.649	1.000	7534.908	1.000	24305.309	1.000
	0.04	0.02	56.267	0.986	1290.017	0.960	6378.824	0.920	18522.688	0.873
	0.08	0.04	51.430	0.942	1043.903	0.863	4412.663	0.765	11039.711	0.674
	0.10	0.05	48.202	0.912	912.452	0.807	3588.589	0.690	8488.141	0.591
2.5	0.00	0.00	35.726	1.000	1311.822	1.000	7335.048	1.000	23950.000	1.000
	0.04	0.02	33.949	0.775	1202.360	0.957	6184.101	0.918	18181.340	0.871
	0.08	0.04	29.521	0.909	958.557	0.855	4224.322	0.759	10707.641	0.669
	0.10	0.05	26.436	0.860	828.042	0.794	3401.540	0.681	8156.134	0.584
3.0	0.00	0.00	8.584	1.000	1203.256	1.000	7090.774	1.000	23515.734	1.000
	0.04	0.02	7.293	0.922	1095.507	0.954	5946.624	0.916	17764.387	0.869
	0.08	0.04	2.760	0.567	854.275	0.843	3994.094	0.751	10301.721	0.662

T A B L E - 8.3

Effects of elastic foundation, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported thin-walled beams (λ= 0, K=0.01, s=2d).

γ	s	d	Values of λ^2 and $q = \lambda / \lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
2	0.00	0.00	113.411	1.000	1574.563	1.000	7906.216	1.000	24952.965	1.000
	0.04	0.02	111.327	0.991	1461.376	0.963	6740.149	0.923	19144.809	0.876
	0.08	0.04	106.146	0.967	1210.974	0.877	4762.502	0.776	11644.762	0.683
	0.10	0.05	102.556	0.951	1077.675	0.827	3935.933	0.706	9092.903	0.604
4	0.00	0.00	161.411	1.000	1622.563	1.000	7954.216	1.000	25000.965	1.000
	0.04	0.02	159.197	0.993	1508.751	0.964	6786.883	0.924	19190.977	0.876
	0.08	0.04	153.472	0.975	1257.070	0.880	4807.706	0.777	11689.590	0.683
	0.10	0.05	149.588	0.963	1123.266	0.832	3980.823	0.707	9137.707	0.605
6	0.00	0.00	241.411	1.000	1702.563	1.000	8034.216	1.000	25080.965	1.000
	0.04	0.02	238.979	0.995	1587.796	0.966	6864.851	0.924	19267.883	0.876
	0.08	0.04	232.365	0.981	1333.911	0.885	4883.068	0.780	11764.303	0.685
	0.10	0.05	227.977	0.972	1199.249	0.839	4055.624	0.710	9212.363	0.606
8	0.00	0.00	353.411	1.000	1814.563	1.000	8146.216	1.000	25192.965	1.000
	0.04	0.02	350.677	0.996	1698.340	0.967	6973.980	0.925	19375.590	0.877
	0.08	0.04	342.828	0.985	1441.493	0.891	4988.573	0.782	11868.906	0.686
	0.10	0.05	337.704	0.978	1305.604	0.848	4160.333	0.715	9316.879	0.608
10	0.00	0.00	497.411	1.000	1958.563	1.000	8290.217	1.000	25336.965	1.000
	0.04	0.02	493.929	0.996	1840.202	0.969	7113.926	0.926	19513.777	0.878
	0.08	0.04	484.814	0.987	1579.773	0.898	5124.179	0.786	12003.357	0.688
	0.10	0.05	478.771	0.981	1442.318	0.858	4294.930	0.720	9451.234	0.611
12	0.00	0.00	673.411	1.000	2134.563	1.000	8466.217	1.000	25512.965	1.000
	0.04	0.02	669.471	0.997	2013.930	0.971	7285.458	0.928	19683.035	0.878
	0.08	0.04	658.378	0.988	1748.797	0.905	5289.933	0.791	12167.723	0.691
	0.10	0.05	651.147	0.983	1609.352	0.868	4459.380	0.726	9615.397	0.614

TABLE - 8.4

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the first and second mode torsional frequencies (first set) of simply supported short thin-walled beams (K=0.01. s=2d).

Δ	s	d	Values of λ^2, I Mode				Values of λ^2, II Mode			
			γ = 0	4	8	12	γ = 0	4	8	12
0.0	0.00	0.00	97.411	161.411	353.411	673.411	1558.563	1622.563	1814.563	2134.563
	0.04	0.02	95.559	159.197	350.677	669.471	1445.771	1508.751	1698.340	2013.930
	0.08	0.04	90.361	153.472	342.827	658.378	1195.602	1257.070	1441.493	1748.797
	0.10	0.05	86.882	149.588	337.704	651.147	1062.477	1123.266	1305.604	1609.352
1.0	0.00	0.00	87.541	151.541	343.541	663.541	1519.085	1583.085	1775.085	2095.085
	0.04	0.02	85.784	149.421	340.902	659.693	1406.831	1469.809	1659.398	1974.985
	0.08	0.04	80.626	143.735	333.092	648.645	1157.668	1219.141	1403.570	1710.881
	0.10	0.05	77.213	139.921	328.039	641.484	1024.978	1085.768	1268.116	1571.888
2.0	0.00	0.00	57.932	121.932	313.932	633.932	1400.649	1464.649	1656.649	1976.649
	0.04	0.02	56.267	119.904	311.383	630.172	1290.017	1352.993	1542.576	1858.151
	0.08	0.04	51.430	114.541	303.900	619.453	1043.903	1105.378	1289.820	1597.157
	0.10	0.05	48.202	110.912	299.033	612.487	912 .452	973.256	1155.642	1459.477
3.0	0.00	0.00	8.584	72.584	264.584	584.584	1203.256	1267.256	1459.256	1779.256
	0.04	0.02	7.293	70.481	262.407	581.189	1095.507	1158.481	1348.052	1663.609
	0.08	0.04	2.760	65.873	255.233	570.791	854.275	915.757	1100.220	1407.593
	0.10	0.05	0.000	62.552	250.682	564.149	-	785.684	968. 132	1272.070

T A B L E - 8.5

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the third and fourth mode torsional frequencies (first set) of simply supported short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, III Mode (γ = 0)	4	8	12	Values of λ^2, IV Mode (γ = 0)	4	8	12
0.0	0.00	0.00	7890.216	7954.216	8146.216	8466.217	24936.965	25000.965	25192.965	25512.965
	0.04	0.02	6724.678	6786.883	6973.980	7285.458	19129.629	19190.977	19375.590	19683.035
	0.08	0.04	4747.425	4807.706	4988.573	5289.933	11629.818	11689.590	11868.906	12167.723
	0.10	0.05	3920.978	3980.823	4160.333	4459.380	9077.973	9137.707	9316.879	9615.397
1.0	0.00	0.00	7801.389	7865.389	8057.389	8377.389	24779.051	24843.051	25035.051	25355.051
	0.04	0.02	6638.099	6700.221	6887.314	7198.785	18977.887	19039.234	19223.910	19531.270
	0.08	0.04	4663.736	4724.018	4904.900	5206.283	11482.309	11542.084	11721.424	12020.270
	0.10	0.05	3837.896	3897.752	4077.286	4376.382	8390.559	8990.305	9169.510	9468.094
2.0	0.00	0.00	7534.908	7598.908	7790.908	8110.908	24305.309	24369.309	24561.309	24881.309
	0.04	0.02	6378.824	6441.023	6628.103	6939.553	18522.688	18584.102	18768.758	19076.086
	0.08	0.04	4412.663	4472.961	4653.874	4955.313	11039.711	11099.510	11278.904	11577.842
	0.10	0.05	3588.589	3648.473	3828.089	4127.317	8488.141	8547.924	8727.244	9026.012
3.0	0.00	0.00	7090.774	7154.774	7346.774	7666.774	23515.734	23579.734	23771.734	24091.734
	0.04	0.02	5946.624	6008.814	6195.874	6507.368	17764.387	17825.793	18010.414	18317.836
	0.08	0.04	3994.094	4054.407	4235.383	4536.924	10301.721	10361.549	10541.033	10840.133
	0.10	0.05	-	3232.762	3412.513	3711.965	-	7810.000	7989.504	8283.578

TABLE - 8.6

Effects of axial compressive load, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported thin-walled beams ($\gamma = 0$, K=0.01, s=2d).

Δ	s	d	Values of λ^2 and $q = \lambda/\lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0.0	0.00	0.00	249.614	1.000	3993.813	1.000	20218.664	1.000	63900.938	1.000
	0.04	0.02	243.820	0.988	3642.962	0.955	16690.797	0.909	46820.211	0.856
	0.08	0.04	227.685	0.955	2962.263	0.856	11414.037	0.751	27857.102	0.660
	0.10	0.05	217.290	0.933	2572.443	0.803	9390.227	0.681	21881.023	0.585
2.0	0.00	0.00	200.266	1.000	3796.419	1.000	19774.527	1.000	63111.367	1.000
	0.04	0.02	194.088	0.984	3439.561	0.952	16216.939	0.906	45940.109	0.853
	0.08	0.04	176.847	0.940	2706.261	0.844	10864.486	0.741	26763.426	0.651
	0.10	0.05	165.634	0.909	2341.124	0.785	8792.682	0.667	20658.918	0.572
4.0	0.00	0.00	52.221	1.000	3204.241	1.000	18442.125	1.000	60742.649	1.000
	0.04	0.02	44.890	0.927	2829.545	0.940	14796.113	0.896	43300.180	0.844
	0.08	0.04	24.331	0.683	2046.722	0.799	9220.086	0.707	23502.168	0.622
	0.10	0.05	10.695	0.453	1648.723	0.717	7013.864	0.617	17055.133	0.530

TABLE - 8.7

Effects of elastic foundation, longitudinal inertia, and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported short thin-walled beams ($\Delta = 0$, $K=0.01$, $s=2d$).

γ	s	d	Values of λ^2 and $q = \lambda / \lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0	0.00	0.00	249.614	1.000	3993.813	1.000	20218.664	1.000	63900.938	1.000
	0.04	0.02	243.820	0.988	3642.962	0.955	16690.797	0.909	46820.211	0.856
	0.08	0.04	227.685	0.955	2926.263	0.856	11414.037	0.751	27857.102	0.660
	0.10	0.05	217.290	0.933	2572.443	0.803	9390.227	0.681	21881.023	0.585
4	0.00	0.00	313.614	1.000	4057.813	1.000	20282.664	1.000	63964.938	1.000
	0.04	0.02	307.523	0.990	3705.920	0.956	16753.008	0.909	46881.867	0.856
	0.08	0.04	290.666	0.963	2987.917	0.858	11475.691	0.752	27920.129	0.661
	0.10	0.05	279.870	0.945	2633.897	0.806	9452.793	0.683	21946.465	0.586
8	0.00	0.00	505.614	1.000	4249.813	1.000	20474.664	1.000	64156.938	1.000
	0.04	0.02	498.630	0.993	3894.979	0.957	16939.648	0.910	47066.828	0.857
	0.08	0.04	479.608	0.974	3172.854	0.864	11660.617	0.755	28109.188	0.662
	0.10	0.05	467.568	0.962	2818.238	0.814	9640.490	0.686	22142.805	0.587
12	0.00	0.00	825.614	1.000	4569.813	1.000	20794.664	1.000	64476.938	1.000
	0.04	0.02	817.143	0.995	4209.766	0.960	17250.523	0.911	47375.094	0.857
	0.08	0.04	794.477	0.981	3481.041	0.873	11968.803	0.759	28424.301	0.664
	0.10	0.05	780.330	0.972	3125.369	0.827	9953.275	0.692	22470.098	0.590

TABLE - 8.8

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the first and second mode torsional frequencies (first set) of clamped-simply supported short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, I Mode				Values of λ^2, II Mode			
			γ 0	4	8	12	γ 0	4	8	12
0.0	0.00	0.00	249.614	313.614	505.614	825.614	3993.813	4057.813	4249.813	4569.813
	0.04	0.02	243.820	307.523	498.630	817.143	3642.962	3705.920	3894.979	4209.766
	0.08	0.04	227.685	290.666	479.608	794.477	2926.263	2987.917	3172.854	3481.041
	0.10	0.05	217.290	279.870	467.568	780.330	2572.443	2633.897	2818.238	3125.369
2.0	0.00	0.00	200.266	264.266	456.266	776.266	3796.419	3860.419	4052.419	4372.420
	0.04	0.02	194.088	257.790	448.898	767.410	3439.562	3502.519	3691.578	4006.365
	0.08	0.04	176.847	239.827	428.758	743.626	2706.261	2767.915	2952.817	3260.980
	0.10	0.05	165.634	228.210	415.907	728.660	2341.124	2042.555	2586.824	2893.826
4.0	0.00	0.00	52.221	116.221	308.221	628.221	3204.241	3268.241	3460.241	3780.241
	0.04	0.02	44.890	108.592	299.700	618.212	2829.545	2892.503	3081.375	3396.348
	0.08	0.04	24.331	87.300	276.242	591.099	2046.722	2108.340	2293.136	2601.243
	0.10	0.05	10.695	73.266	260.949	573.678	1648.723	1610.082	1894.137	2200.772

T A B L E - 8.9

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the third and fourth mode torsional frequencies (first set) of clamped-simply supported short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, III Mode: γ = 0	γ = 4	γ = 8	γ = 12	Values of λ^2, IV Mode: γ = 0	γ = 4	γ = 8	γ = 12
0.0	0.00	0.00	20218.664	20282.664	20474.664	20794.664	63900.938	63964.938	64156.938	64476.938
	0.04	0.02	16690.767	16753.008	16939.648	17250.523	46820.211	46881.867	47066.828	47375.094
	0.08	0.04	11414.037	11475.691	11660.617	11968.803	27857.102	27920.129	28109.188	28424.301
	0.10	0.05	9390.227	8452.793	9640.490	9953.275	21881.023	21946.465	22142.805	22470.098
2.0	0.00	0.00	19774.527	19838.527	20030.527	20350.527	63111.367	63175.367	63367.367	63687.367
	0.04	0.02	16216.939	16279.152	16465.789	16776.852	45940.109	46001.766	46186.727	46494.805
	0.08	0.04	10864.486	10926.117	11110.938	11418.973	26763.426	26826.336	27015.141	27329.809
	0.10	0.05	8792.682	8855.143	9042.502	9354.711	20658.918	20724.051	20919.469	21245.180
4.0	0.00	0.00	18442.125	18506.125	18698.125	19018.125	60742.649	60806.649	60998.649	61318.649
	0.04	0.02	14796.113	14858.139	15044.775	15355.838	43300.180	43361.836	43546.797	43855.063
	0.08	0.04	9220.086	9281.611	9466.164	9773.723	23502.168	23564.844	23752.856	24066.199
	0.10	0.05	7013.864	7076.006	7262.415	7573.032	17055.133	17119.438	17312.328	17633.840

TABLE - 8.10

Effects of axial compressive load, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped short thin-walled beams (γ= 0, K=0.01, s=2d).

Δ	s	d	Values of λ^2 and $q = \lambda/\lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0.0	0.00	0.00	519.521	1.000	8312.322	1.000	42081.117	1.000	132997.094	1.000
	0.04	0.02	506.516	0.987	7553.774	0.953	34643.352	0.907	97904.031	0.858
	0.08	0.04	472.111	0.953	6119.002	0.858	24856.652	0.769	66324.172	0.706
	0.10	0.05	450.494	0.931	5463.667	0.811	21719.863	0.718	66035.985	0.705
2.0	0.00	0.00	466.883	1.000	8101.770	1.000	41607.375	1.000	132154.875	1.000
	0.04	0.02	452.002	0.984	7313.990	0.950	34029.055	0.904	96638.172	0.855
	0.08	0.04	412.165	0.940	5802.740	0.846	23865.852	0.757	63592.719	0.694
	0.10	0.05	386.737	0.910	5093.349	0.793	20378.367	0.700	60305.024	0.676
4.0	0.00	0.00	308.969	1.000	7470.111	1.000	40186.141	1.000	129628.250	1.000
	0.04	0.02	288.338	0.966	6594.636	0.940	32187.020	0.895	92843.719	0.846
	0.08	0.04	232.373	0.867	4857.074	0.806	20935.410	0.722	55818.211	0.656
	0.10	0.05	195.653	0.796	3994.760	0.731	16570.820	0.642	44487.195	0.586

TABLE - 8.11

Effects of elastic foundation, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped short thin-walled beams ($\Delta = 0$, K=0.01, s=2d).

γ	s	d	Values of λ^2 and $q = \lambda/\lambda_o$							
			I Mode	q_1	II Mode	q_2	III Mode	q_3	IV Mode	q_4
0	0.00	0.00	519.521	1.000	8312.322	1.000	42081.117	1.000	132997.094	1.000
	0.04	0.02	506.516	0.987	7553.774	0.953	34643.352	0.907	97904.031	0.858
	0.08	0.04	472.111	0.953	6119.002	0.858	24856,652	0.769	66324.172	0.706
	0.10	0.05	450.494	0.931	5463.663	0.811	21719.863	0.718	66035.985	0.705
4	0.00	0.00	583.521	1.000	8376.322	1.000	42145.117	1.000	133061.094	1.000
	0.04	0.02	570.218	0.989	7616.856	0.954	34706.063	0.907	97966.360	0.858
	0.08	0.04	535.162	0.958	6181.943	0.859	24923.539	0.769	66404.719	0.706
	0.10	0.05	513.281	0.938	5527.894	0.812	21795.211	0.719	65822.531	0.703
8	0.00	0.00	775.521	1.000	8568.322	1.000	42337.117	1.000	133253.094	1.000
	0.04	0.02	761.326	0.991	7805.977	0.954	34893.945	0.908	98155.735	0.858
	0.08	0.04	724.305	0.966	6370.738	0.862	25124.254	0.770	66646.406	0.707
	0.10	0.05	701.615	0.951	5720.594	0.817	22021.434	0.721	65231.055	0.700
12	0.00	0.00	1095.521	1.000	8888.322	1.000	42657.117	1.000	133573.094	1.000
	0.04	0.02	1079.714	0.993	8121.261	0.956	35207.117	0.908	98470.391	0.859
	0.08	0.04	1039.543	0.974	6685.378	0.867	25458.801	0.773	67049.672	0.708
	0.10	0.05	1015.444	0.963	6041.766	0.824	22399.109	0.725	64369.406	0.694

T A B L E - 8.12

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the first and second mode torsional frequencies (first set) of clamped-clamped short thin-walled beams (K=0.01, s=2d).

Δ	s	d	Values of λ^2, I Mode				Values of λ^2, II Mode			
			γ 0	4	8	12	γ 0	4	8	12
0.0	0.00	0.00	519.521	583.521	775.521	1095.521	8312.322	8376.322	8568.322	8388.322
	0.04	0.02	506.516	570.218	761.326	1079.714	7553.774	7616.856	7805.977	8121.261
	0.08	0.04	472.111	535.162	724.305	1039.543	6119.002	6181.943	6370.738	6685.378
	0.10	0.05	450.494	513.281	701.615	1015.444	5463.663	5527.894	5720.594	6041.766
2.0	0.00	0.00	466.883	530.883	722.883	1042.333	8101.770	8165.770	8357.770	8677.770
	0.04	0.02	452.002	515.580	706.688	1025.076	7313.990	7376.947	7566.192	7881.476
	0.08	0.04	412.165	475.208	664.344	979.558	5802.740	5865.620	6054.274	6368.666
	0.10	0.05	386.737	449.511	637.808	951.564	5093.349	5157.397	5349.538	5669.769
4.0	0.00	0.00	308.969	372.969	564.969	884.969	7470.111	7534.111	7726.111	8046.111
	0.04	0.02	288.338	351.916	543.148	861.536	6594.636	6657.594	6846.839	7161.998
	0.08	0.04	232.373	295.400	484.505	799.657	4857.074	4919.814	5108.020	5421.674
	0.10	0.05	195.655	258.385	446.558	760.114	3994.760	4058.287	4248.851	4566.443

TABLE - 8.13

Combined effects of axial compressive load and elastic foundation in combination with longitudinal inertia and shear deformation on the third and fourth mode torsional frequencies (first set) of clamped-clamped short thin-walled beams (K=0.01, s=2d).

Δ	s	d	III Mode, γ = 0	III Mode, γ = 4	III Mode, γ = 8	III Mode, γ = 12	IV Mode, γ = 0	IV Mode, γ = 4	IV Mode, γ = 8	IV Mode, γ = 12
			Values of λ^2, III Mode				Values of λ^2, IV Mode			
0.0	0.00	0.00	42081.117	42145.117	42337.117	42657.117	132997.094	133061.094	133253.094	133573.09
	0.04	0.02	34643.352	34706.063	34893.945	35207.117	97904 .031	97966.860	98155.735	98470.39
	0.08	0.04	24856.652	24923.539	25124.254	25458.801	66324.172	66404 .719	66646.406	67049.67
	0.10	0.05	21719.863	21795.211	22021.434	22399.109	66035.985	65822.531	65231.055	64369.40
2.0	0.00	0.00	41607.375	41671.375	41863.375	42183.375	132154.875	132218.875	132410.875	132730.87
	0.04	0.02	34029.055	34091.766	34279.641	34592.938	96638.172	96701.125	96889.750	97204.54
	0.08	0.04	23865.852	23932.473	24132.395	24465.621	63592.719	63671.820	63909.094	64304.89
	0.10	0.05	20378.367	20452.328	20674.352	21044.898	60305.024	60477.586	61004.914	61920.90
4.0	0.00	0.00	40186.141	40250.141	40442.141	40762.141	129628.250	129692.250	129884.250	125993.18
	0.04	0.02	32187.020	32249.606	32437.484	32750.656	92843.719	92906.672	93095.297	87095.59
	0.08	0.04	20935.410	21001.320	21198.992	21528.492	55818.211	55893.649	56120.031	56497.59
	0.10	0.05	16570.820	16641.430	16853.352	17206.852	44487.195	44583.672	44873.789	45359.73

TABLE - 8.14

Effects of axial compressive load, elastic foundation and warping in combination with longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of simply supported short thin-walled beams (s=0.10 and d=0.05).

Δ	γ	Values of $q = \lambda/\lambda_o$ for K=1.0				Values of $q = \lambda/\lambda_o$ for K=10.0			
		I Mode	II Mode	III Mode	IV Mode	I Mode	II Mode	III Mode	IV Mode
0.0	0	0.9487	0.8297	0.7084	0.6063	0.9973	0.9847	0.9572	0.9176
	4	0.9643	0.8357	0.7108	0.6075	0.9973	0.9848	0.9573	0.9177
	8	0.9779	0.8511	0.7178	0.6110	0.9976	0.9851	0.9577	0.9180
	12	0.9834	0.8703	0.7287	0.6167	0.9976	0.9855	0.9582	0.9185
1.5	0	0.9377	0.8203	0.7005	0.5996	0.9974	0.9845	0.9569	0.9170
	4	0.9604	0.8272	0.7031	0.6008	0.9974	0.9846	0.9570	0.9171
	8	0.9771	0.8444	0.7105	0.6045	0.9976	0.9849	0.9573	0.9175
	12	0.9832	0.8656	0.7220	0.6104	0.9977	0.9854	0.9579	0.9180
3.0	0	0.7180	0.7832	0.6734	0.4776	0.9974	0.9841	0.9557	0.9152
	4	0.9359	0.7937	0.6766	0.5790	0.9974	0.9842	0.9559	0.9154
	8	0.9740	0.8191	0.6857	0.5831	0.9976	0.9845	0.9562	0.9157
	12	0.9825	0.8496	0.6995	0.5898	0.9977	0.9850	0.9568	0.9162

TABLE - 8.15

Effects of axial compressive load, elastic foundation and warping in combination with longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-simply supported short thin-walled beams (s=0.10 and d=0.05).

Δ	γ	Values of $q = \lambda / \lambda_o$ for K=1.0				Values of $q = \lambda / \lambda_o$ for K=10.0			
		I Mode	II Mode	III Mode	IV Mode	I Mode	II Mode	III Mode	IV Mode
0	0	0.9330	0.8026	0.6815	0.5852	1.0091	0.9729	0.9001	0.8461
	4	0.9447	0.8057	0.6827	0.5857	1.0083	0.9730	0.9004	0.8465
	8	0.9616	0.8143	0.6862	0.5875	1.0062	0.9733	0.9011	0.8476
	12	0.9722	0.8270	0.6918	0.5903	1.0035	0.9738	0.9025	0.8495
2	0	0.9094	0.7853	0.6668	0.5721	1.0086	0.9699	0.8941	0.8347
	4	0.9293	0.7889	0.6681	0.5727	1.0077	0.9700	0.8944	0.8350
	8	0.9547	0.7990	0.6719	0.5746	1.0056	0.9703	0.8952	0.8361
	12	0.9689	0.8135	0.6780	0.5776	1.0030	0.9709	0.8967	0.8378
4	0	0.4526	0.7173	0.6167	0.5299	1.0067	0.9598	0.8754	0.8032
	4	0.7940	0.7234	0.6184	0.5306	1.0059	0.9600	0.8757	0.8035
	8	0.9201	0.7399	0.6232	0.5327	1.0038	0.9606	0.8767	0.8045
	12	0.9556	0.7630	0.6310	0.5363	1.0013	0.9615	0.8784	0.8061

T A B L E - 8.16

Effects of axial compressive load, elastic foundation and warping in combination with longitudinal inertia and shear deformation on the first four torsional frequencies (first set) of clamped-clamped short thin-walled Beams (s=0.10 and d=0.05).

		Values of $q = \lambda / \lambda_o$ for K=1.0				Values of $q = \lambda / \lambda_o$ for K=10.0			
Δ	γ	I Mode	II Mode	III Mode	IV Mode	I Mode	II Mode	III Mode	IV Mode
0	0	0.9358	0.8150	0.7230	0.6891	1.0094	0.9991	0.9632	0.9177
	4	0.9418	0.8166	0.7237	0.6882	1.0090	0.9991	0.9632	0.9177
	8	0.9539	0.8212	0.7258	0.6855	1.0080	0.9990	0.9633	0.9178
	12	0.9645	0.8284	0.7292	0.6813	1.0066	0.9988	0.9635	0.9180
2	0	0.9159	0.7975	0.7045	0.6874	1.0091	0.9980	0.9615	0.9156
	4	0.9250	0.7992	0.7052	0.6884	1.0088	0.9979	0.9615	0.9157
	8	0.9424	0.8044	0.7074	0.6913	1.0077	0.9979	0.9616	0.9158
	12	0.9572	0.8124	0.7109	0.6966	1.0064	0.9973	0.9618	0.9159
4	0	0.8104	0.7370	0.6471	0.5919	1.0083	0.9944	0.9561	0.9094
	4	0.8428	0.7395	0.6479	0.5924	1.0079	0.9944	0.9562	0.9094
	8	0.8944	0.7469	0.6505	0.5939	1.0068	0.9944	0.9563	0.9095
	12	0.9296	0.7584	0.6546	0.5964	1.0055	0.9943	0.9565	0.9097

reductions in the torsional frequencies due to increase in the axial compressive load can be observed from these tables to be slightly higher than those when the effects are neglected.

The combined effect of elastic foundation, longitudinal inertia and shear deformation on the first four torsional frequencies (first set) are shown in Tables 8.3, 8.7 and 8.11 for values of K = 0.01 and s = 2d. From these results it can be noted that the percentage increase in the torsional frequencies due to elastic foundation is slightly more than those when the second order effects are neglected. The results presented in Tables 8.4, 8.5, 8.8, 8.9, 8.12 and 8.13 show the combined effects of axial compressive load and elastic foundation in combination with the effects of longitudinal inertia and shear deformation on the first and second, third and fourth torsional frequencies (first set) of simply supported, clamped-clamped and clamped-simply supported beams respectively. It can be observed from these tables that the combined effects are almost the algebroic sum of the individual influences of various effects on the torsional frequencies of vibration. The results for the modifying quotients for the first four modes of vibration for simply-supported, clamped-clamped, and clamped-simply supported beams are respectively presented in Tables 8.14, 8.15 and 8.16 for values of s = 0.10, d = 0.05 and for various values of $\triangle$, γ and K. From these results we observe that for any set of values of K and γ , the influence of increase in the values of $\triangle$ in the range 0.0 to 3.0 is to decrease the modifying quotients (i.e., to increase the second order effects on the frequencies of vibration) for various modes by about 25 percent. For any constant set of values of Δ and K, the effect of increase in the values of γ in the range 0 to 12 is to increase the modifying quotients (i.e., to decrease the second order effects on the frequencies of vibration) for various modes at the most by 15 percent. For constant values of Δ and γ, the effect of increasing the value of K from 1.0 to 10.0 is to increase the modifying quotients for various modes by about 10 percent.

It is also observed that, for constant values of K and γ, the reduction in the frequency of vibration at the first mode is quite considerable for values of Δ nearing Δ_{cr}. From the various results presented in this section, we can conclude that the effects of shear deformation and longitudinal inertia on the torsional frequencies at higher modes become increasingly important for a beam with smaller values of warping parameter K and foundation parameter γ and for larger values of $\Delta \leq \Delta_{cr}$.

CHAPTER - IX

FINITE ELEMENT ANALYSIS OF TORSIONAL VIBRATIONS AND STABILITY OF SHORT THIN-WALLED BEAMS RESTING ON CONTINUOUS ELASTIC FOUNDATION*.

9.1. INTRODUCTION:

The problem of torsional vibrations and stability of lengthy thin-walled beams of open section resting on Winkler-type elastic foundation is solved in Chapter III utilizing finite-element method. The stiffness, stability and mass matrices derived therein, does not include the second order effects such as longitudinal inertia and shear deformation. These second order effects cannot be neglected in the case of short and deep thin-walled beams and, as is shown in Chapter IV, they drastically change the torsional frequencies at higher modes of vibration.

The present chapter, therefore, aims at extending the finite element method presented in Chapter III to include the effects of longitudinal inertia and shear deformation. New stiffness, stability coefficient and mass matrices for a short or deep thin-walled beam are developed in this Chapter, which include the effects of longitudinal inertia and shear deformation in addition to the effects of axial time-invariant compressive load and elastic foundation. The method developed herein

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* A paper by the author based on the results from this Chapter is communicated to Journal of Applied Mechanics, Transactions of ASME, for publication. See Ref.(56)

is useful in analyzing both uniform and non-uniform beams with any complex boundary conditions. The new stiffness and stability coefficient matrices are made use of in conjunction with the consistant mass matrix for finding the torsional frequencies, buckling loads and mode shapes of short uniform thin-walled beams with various end conditions. Results obtained for the case of a simply supported beam by the finite element method are compared with the exact ones obtained in Chapter VIII and an excellent agreement is observed even for a coarse sub-division of the beam.

9.2. MODIFIED STRAIN ENERGY EXPRESSION INCLUDING THE EFFECTS OF AXIAL LOAD AND ELASTIC FOUNDATION:

Substituting Eq.(5.1) into Eq.(8.1), the strain energy U_4, due to the Winkler-type elastic foundation can be written in a modified form as:

$$U_4 = \frac{1}{2} \int_0^L K_t(\phi_t + \phi_s)^2 \, dz \tag{9.1}$$

Utilizing Eqs.(5.14) and (9.1), the total strain energy U at any instant t including the effect of Winkler-type elastic foundation can be written in a modified form as:

$$\begin{aligned} U &= U_1 + U_2 + U_3 + U_4 \\ &= \frac{1}{2} \int_0^L \Big[GC_s\Big(\frac{\partial\phi_t}{\partial z} + \frac{\partial\phi_s}{\partial z}\Big)^2 + EC_w\Big(\frac{\partial^2\phi_t}{\partial z^2}\Big)^2 \\ &\quad + K' A_f G\frac{h^2}{2}\Big(\frac{\partial\phi_s}{\partial z}\Big)^2 + K_t(\phi_t + \phi_s)^2 \Big] dz \end{aligned} \tag{9.2}$$

Substituting Eq.(5.1) into Eq.(8.3) the potential energy, W, due to the time-invariant axial compressive load P can be written in a modified form as:

$$W = \frac{1}{2} \int_0^L \frac{PI_p}{A} \left(\frac{\partial \phi_t}{\partial z} + \frac{\partial \phi_s}{\partial z}\right)^2 dz \qquad (9.3)$$

The total kinetic energy, T_k, at any time t in the modified form is given by:

$$T_k = \frac{1}{2} \int_0^L \left[\rho I_p \left(\frac{\partial \phi_t}{\partial t} + \frac{\partial \phi_s}{\partial t}\right)^2 + \rho C_w \left(\frac{\partial^2 \phi_t}{\partial z \partial t}\right)^2 \right] dz \qquad (9.4)$$

which is same as Eq.(5.15).

9.3. MODIFIED NATURAL BOUNDARY CONDITIONS:

Except for the case of a free end, the boundary conditions for simply supported and fixed ends remain the same as those given by Eqs.(5.16) and (5.17).

For the case of a ''free end'', the modified natural boundary conditions for the present problem become:

$$\frac{\partial^2 \phi_t}{\partial z^2} = 0; \quad \left(GC_s - \frac{PI_p}{A}\right) \frac{\partial \phi_t}{\partial z} + \left(GC_s - \frac{PI_p}{A} + K' A_f G \frac{h^2}{2}\right) \frac{\partial \phi_s}{\partial z} = 0 \qquad (9.5)$$

9.4. DERIVATION OF ELEMENT MATRICES INCLUDING AXIAL LOAD, ELASTIC FOUNDATION AND SECOND ORDER EFFECTS:

The expressions for the strain energy U, potential energy W and, Kinetic energy T_k, given by Eqs.(9.2), (9.3) and (9.4) respectively, for an element of length, l, can be written as follows:

$$U = \frac{1}{2} \int_0^1 \left[GC_s(\phi_t' + \phi_s')^2 + EC_w(\phi_t'')^2 + K' A_f G \frac{h^2}{2}(\phi_s')^2 + K_t(\phi_t + \phi_s)^2 \right] dz \qquad (9.6)$$

$$W = \frac{1}{2} \int_0^1 \frac{PI_p}{A} (\phi_t' + \phi_s')^2 \, dz \qquad (9.7)$$

and

$$T_k = \frac{1}{2} \int_0^1 \left[\rho I_p(\dot{\phi}_t + \dot{\phi}_s)^2 + \rho C_w(\dot{\phi}_t')^2 \right] dz \qquad (9.8)$$

Direct substitution of Eqs.(5.24) to (5.36) into Eqs.(9.6), (9.7) and (9.8) and the resulting expressions into Hamilton's principle, Eq.(3.34), yields (for the Nth element):

$$\begin{aligned}
\bar{\delta} I_N = \bar{\delta} \int_{t_1}^{t_2} \Bigg\{ & \rho I_p \Bigg[\int_0^1 \dot{\bar{R}}_{tN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{tN} \, dz + \int_0^1 \dot{\bar{R}}_{sN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{sN} \, dz \\
& + \int_0^1 \dot{\bar{R}}_{tN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{sN} dz + \int_0^1 \dot{\bar{R}}_{sN}^T \bar{A}^T \bar{A} \dot{\bar{R}}_{tN} \, dz \Bigg] \\
& + \frac{\rho C_w}{2} \int_0^1 \dot{\bar{R}}_{tN}^T \bar{A}_1^T \bar{A}_1 \dot{\bar{R}}_{tN} \, dz \\
& - \frac{1}{2} \int_0^1 \bar{R}_{tN}^T \left[EC_w \bar{A}_2^T \bar{A}_2 + GC_s \bar{A}_1^T \bar{A}_1 + K_t \bar{A}^T \bar{A} \right] \bar{R}_{tN} \, dz \\
& - \frac{1}{2} \int_0^1 \bar{R}_{sN}^T \left[(GC_s + K' A_f G \, h^2/2) \bar{A}_1^T \bar{A}_1 + K_t \bar{A}^T \bar{A} \right] \bar{R}_{sN} \, dz \\
& - \frac{GC_s}{2} \left[\int_0^1 \bar{R}_{tN}^T \bar{A}_1^T \bar{A}_1 \bar{R}_{sN} \, dz + \int_0^1 \bar{R}_{sN}^T \bar{A}_1^T \bar{A}_1 \bar{R}_{tN} \, dz \right]
\end{aligned}$$

$$- \frac{K_t}{2}\left[\int_0^1 \bar{R}_{tN}^T \bar{A}^T\bar{A}\, \bar{R}_{sN}\, dz + \int_0^1 \bar{R}_{sN}^T \bar{A}^T\bar{A}\, \bar{R}_{tN}\, dz\right]$$

$$+ \frac{PI_p}{2A}\left[\int_0^1 \bar{R}_{tN}^T \bar{A}_1^T \bar{A}_1\, \bar{R}_{tN}\, dz + \int_0^1 \bar{R}_{sN}^T \bar{A}_1^T \bar{A}_1\, \bar{R}_{sN}\, dz\right.$$

$$\left.\left. + \int_0^1 \bar{R}_{tN}^T \bar{A}_1^T \bar{A}_1\, \bar{R}_{sN}\, dz + \int_0^1 \bar{R}_{sN}^T \bar{A}_1^T \bar{A}_1\, \bar{R}_{tN}\, dz\right]\right\} dt$$

$$= 0 \tag{9.9}$$

Eq.(9.9) can be written more concisely as follows:

$$\bar{\delta I}_N = \bar{\delta}\int_{t_1}^{t_2} \frac{1}{2}\left[(\rho I_p L)\,\dot{\bar{q}}_N^T\,\bar{m}_N\,\dot{\bar{q}}_N - (EC_w/L^3)\,\bar{q}_N^T\,\bar{k}_N\,\bar{q}_N\right.$$

$$\left. + (PI_p/AL)\,\bar{q}_N^T\,\bar{s}_N\,\bar{q}_N\right] dt = 0 \tag{9.10}$$

In Eq.(9.10) the terms $(\rho I_p L)\,\bar{m}_N$, $(EC_w/L^3)\bar{k}_N$ and $(PI_p/AL)\bar{s}_N$ denote respectively the mass matrix $\bar{M}_N$, the stiffness matrix $\bar{K}_N$ and stability coefficient matrix $\bar{s}_N$ of the Nth element. The matrices $\bar{m}_N$ and $\bar{q}_N$ obtained herein are the same as Eqs.(5.41) and (5.43) respectively. The matrices $\bar{k}_N$ and $\bar{s}_N$ are as follows:

$$\bar{k}_N = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{21}^T \\ \bar{K}_{21} & \bar{K}_{22} \end{bmatrix} \tag{9.11}$$

where

$$\bar{K}_{11} = \begin{bmatrix} 12N^2 & & \text{Sym.} & \\ 6N & 4 & & \\ -12N^2 & -6N & 12N^2 & \\ 6N & 2 & -6N & 4 \end{bmatrix}$$

$$
+\frac{K^2}{30N^2}\begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix}
$$

$$
+\frac{4\gamma^2}{420N^4}\begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \qquad (9.12)
$$

$$
\bar{K}_{21} = \frac{K^2}{30N^2}\begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix}
$$

$$
+\frac{4\gamma^2}{420N^4}\begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \qquad (9.13)
$$

$$
K_{22} = \frac{(s^2K^2+1)}{30\ s^2N^2}\begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix}
$$

$$
+\frac{4\gamma^2}{420N^4}\begin{bmatrix} 156N^2 & & \text{Sym.} & \\ 22N & 4 & & \\ 54N^2 & 13N & 156N^2 & \\ -13N & -3 & -22N & 4 \end{bmatrix} \qquad (9.14)
$$

and

$$\bar{s}_N = \begin{bmatrix} \bar{s}_{11} & \bar{s}_{21}^T \\ \bar{s}_{21} & \bar{s}_{22} \end{bmatrix} \qquad (9.15)$$

where

$$\bar{s}_{11} = \bar{s}_{21} = \bar{s}_{22} = \begin{bmatrix} 36N^2 & & \text{Sym.} & \\ 3N & 4 & & \\ -36N^2 & -3N & 36N^2 & \\ 3N & -1 & -3N & 4 \end{bmatrix} \qquad (9.16)$$

Following the procedure outlined in Chapters III and V, the equations of motion for the discretized system can now be obtained from Eq.(9.10) as follows:

$$[\bar{k}_N - \triangle^2 \bar{s}_N] [\bar{Q}_N] = \lambda^2 [\bar{m}_N] [\bar{Q}_N] \qquad (9.17)$$

where the non-dimensional parameters $\triangle^2$ and λ^2 are given by Eqs.(3.47) and (3.48).

In a similar way the equations of equilibrium for the totally assembled beam can be obtained as:

$$[\bar{k} - \triangle^2 \bar{s}] [\bar{Q}] = \lambda^2 [\bar{m}] [\bar{Q}] \qquad (9.18)$$

where $\bar{k}$, $\bar{s}$, $\bar{m}$ and $\bar{Q}$ denote the totally assembled matrices corresponding to the element matrices $\bar{k}_N$, $\bar{s}_N$, $\bar{m}_N$ and $\bar{Q}_N$ defined previously.

9.5. RESULTS AND CONCLUSIONS:

Results for the first and second sets of values of λ^2 for various values of the axial load parameter Δ and foundation parameter γ for simply supported beams for values of K = 1.541, s = 0.046 and d = 0.023, are obtained on IBM 1130 Computer at Andhra University, Waltair and are presented in Tables 9.1 and 9.2.

In the case of the first set of frequencies, the values of λ obtained for the first four modes of vibration, for various values of γ and Δ , for a division of the beam into N = 2 and 3 segments are shown in Table 9.1 and are compared with the exact results obtained using the analysis presented in Chapter VIII. For, the second set, the values of λ obtained for the first four modes of vibration for N = 2 and 3 are shown in Table 9.2 and are compared with exact results. The exact results for the first and second sets were obtained using Eq.(8.45).

From Tables 9.1 and 9.2, it can be observed that, for all cases, the results obtained by finite element method even for very coarse subdivisions of the beam, are in excellent agreement with the exact ones. As stiffness and mass matrices including shear deformation and longitudinal inertia in addition to axial load and elastic foundation, involve double the number of degrees of freedom than those that exist if the secondary effects are neglected, twice as many natural frequencies result. In tables 9.1 and 9.2 the lower and higher spectrum of frequencies of simply supported beam are respectively listed. The second set of frequencies can also be observed to be in excellent agreement with the

TABLE - 9.1

Comparison of first set of values of λ for various values of Δ and γ from the Finite Element Method and those from exact analysis given in Chapter-VIII for a Simply Supported beam (K = 1.541, s = 0.046, d = 0.023).

Value of γ	Value of Δ	Mode No.	No. of Elements 2	No. of Elements 3	Exact Results
0.0	3.0	I	12.3586	5.1254	4.7989
		II	33.9722	29.9049	29.7652
		III	101.0481	89.0871	65.9710
		IV	153.1285	142.7591	113.5342
2.0	3.0	I	11.3084	3.9253	3.2886
		II	34.3318	30.3129	23.3118
		III	101.1685	89.2232	65.4434
		IV	153.2073	142.8436	111.9216
2.0	0.0	I	23.2132	11.1546	10.2442
		II	42.5088	39.2334	35.7593
		III	108.1488	97.0513	73.8721
		IV	161.4194	151.3481	139.3192
4.0	3.0	I	8.4977	4.8672	4.3795
		II	35.1243	31.2071	25.9475
		III	101.4378	89.5272	63.8066
		IV	153.3832	143.0309	110.1274

TABLE - 9.2

Comparison of Second set of values of λ for various values of Δ and γ from the Finite Element Method and those from exact analysis given in Chapter - VIII for a Simply Supported beam (K = 1.541, s = 0.046, d = 0.023).

Value of γ	Value of Δ	Mode No.	No. of Elements: 2	No. of Elements: 3	Exact Results.
0.0	2.0	I	962.7403	960.9861	842.969
		II	1006.2539	999.3401	874.078
		III	1093.2914	1071.8298	922.431
		IV	1191.2887	1164.5545	984.441
2.0	3.0	I	962.7403	960.9873	842.969
		II	1006.2539	999.3391	874.078
		III	1093.9256	1071.8298	922.427
		IV	1191.2887	1164.5545	984.433
2.0	0.0	I	962.7414	960.9839	842.970
		II	1006.2596	999.3436	874.081
		III	1093.3223	1071.8504	922.442
		IV	1191.3344	1164.6002	984.467
4.0	3.0	I	962.7403	960.9861	842.969
		II	1006.2539	999.3402	874.079
		III	1093.2937	1071.8309	922.432
		IV	1191.2887	1164.5545	984.442

exact ones. In Chapters IV and VIII these second set of frequencies are discussed in detail.

As was mentioned previously, results for other boundary conditions can be easily obtained using the above stiffness and mass matrices with suitable changes in the Computer program and the data. The advantage of using the finite element method is that a beam with non-uniform section can also be analyzed by deviding the beam into a number of segments and assuming each segment has a constant cross section. This method provides us with an upper bound to the exact frequencies of the system and is quite general, satisfactorily encompassing all boundary conditions.

CHAPTER - X

NON-LINEAR TORSIONAL STABILITY OF LENGTHY THIN-WALLED BEAMS OF OPEN SECTION RESTING ON CONTINUOUS ELASTIC FOUNDATION*.

10.1. INTRODUCTION:

It is not uncommon, in structural design, to regard the elastic buckling load of a slender structural member as its failure load, and to pay little attention to its post-buckling behaviour. However, some structural members, such as simply supported thin plates loaded in compression, can support loads significantly greater than their elastic critical loads without deflecting excessively. This reserve of strength after buckling is due mainly to a redistribution of stress from the more flexible central area of the plate to the unloaded-edge regions (13). On the other hand, the load carrying capacity of some thin shell structures reduces rapidly after buckling. Such a structure is extremely sensitive to imperfections and disturbances, and may deform excessively at loads much less than its elastic critical load (45). Clearly, the post buckling behaviour of a structural member may have a decisive influence on the relation between its buckling and ultimate strengths.

The classical linear buckling theories (99) for elastic beams and columns necessarily predict buckling at loads that remain constant as the buckling amplitudes increase. Euler (99) first investigated the elastic flexural post-buckling behaviour of columns in 1744, by using the exact expression for curvature instead of the familiar small deflection approximation. This resulted in a post-buckling curve that rises so slowly that there is no significant increase in the load-carrying capacity until the deformations become gross.

The non-linear behaviour of members in uniform torsion was first investigated by Young (102) who considered circular cross sections. A related problem, the torsional stiffness of narrow rectangular sections under uniform axial tension, was examined by Buckley (14) and Weber (102) investigated the non-linear behaviour of narrow rectangular strips in pure torsion. Later, Cullimore (21) studied the behaviour of thin-walled I and Z sections. Weber and Cullimore showed that the torsional stiffness increases with the twist, and that this is due to a system of stresses acting along the helical fibres of the twisted member. The stress system is self equilibrating so that the outer fibres are in tension and the fibres closer to the twist axis are in compression.

Although Cullimore correctly derived the result for narrow rectangular members his expression for the non-linear torque component for I and Z sections is in doubt, because he used a constant lever arm, to obtain the torque contributed by the flange, instead of a variable lever arm, which is the distance from the twist axis to any point on the flange. Furthermore, his assumption of very thin walls leads to some inaccuracies when applied to the I and Z sections in common use. A more accurate theory of non-linear non-uniform torsion of thin-walled beams of open section is presented by Tso and Ghobarah (105) using the principle of minimum potential energy. Their theory takes into account the effect of large torsional deformation and allows very general loading and boundary conditions.

It can be seen that there is a surprising paucity of work on the elastic torsional post-buckling behaviour of doubly symmetric beams, in comparison with the extensive work on other structures (45). In particular, the behaviour of simply-supported and clamped beams and of I-section members resting on continuous elastic foundation has not been investigated. The purpose of the present Chapter, then, is to study theoretically the elastical torsional post-buckling behaviour of statically determinate beams of I-section resting on continuous Winkler type elastic foundation.

10.2. DEVELOPMENT OF GOVERNING DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS:

Consider a thin-walled beam of doubly-symmetric open cross section subject to axial compressive load. The relationship between the total torque T_t and the corresponding angle of twist $\emptyset$ in pure elastic torsion of a uniform thin-walled beam is given by Saint-Venant as:

$$T_t = GC_s \frac{d\emptyset}{dz} \tag{10.1}$$

In the case of non-uniform torsion, Eq.(10.1) is extended to allow for the warping of the cross-sections of the beam; and

$$T_t = GC_s \frac{d\phi}{dz} - EC_w \frac{d^3\phi}{dz^3} \qquad (10.2)$$

The above Eq.(10.2) gives reasonable results for angles of twist approximately no greater than 5^0.

Experimental results obtained by Goodier (38) from tests have shown good qualitative, but poor quantitative, agreement with the theoretical conclusions from Eq.(10.2). If one examines the work of Weber (102), Gregory (42), Terrington (97) and Tso and Ghobarah (105), it can be seen that Eq.(10.2) is not complete insofar as there is a further torque component term to be considered. This term is due to the 'shortening effect' arising from torsion, described by Weber (102) and allowed for by Gregory (42) and, Tso and Ghobarah (105). Allowing for this component of torque, Eq.(10.2), becomes

$$T_t = GC_s \frac{d\phi}{dz} - EC_w \frac{d^3\phi}{dz^3} + 2EF\left(\frac{d\phi}{dz}\right)^3 \qquad (10.3)$$

where F is a constant dependent on cross sectional properties and is defined by

$$F = I_R - (I_{pc}/A)^2 \qquad (10.4)$$

in which I_{pc} is half the polar moment of inertia about the shear center and I_R the fourth moment of inertia about the shear center.

In the case of a thin-walled doubly symmetric I-beam of flange and web thicknesses t_f and t_w respectively; height between the centerlines of the flanges h, flange width b_f, and flange and web thicknesses being assumed as small compared with height h, i.e.

$t_f << h$, and $t_w << h$, the geometric properties in Eq.(10.4) can be evaluated as follows (105):

$$I_{R} = \frac{h^5 t_w}{320} + \frac{bh^4 t_f}{32} + \frac{b^5 t_f}{160} + \frac{b^3 h^2 t_f}{48} \qquad (10.5)$$

and

$$I_{pc} = (1/24)\,(h^3 t_w + 2b^3 t_f + 6bh^2 t_f) \qquad (10.6)$$

For a beam resting on a continuous Winkler type elastic foundation and subjected to an axial compressive load P, we have

$$\frac{dT_t}{dz} = \frac{PI_p}{A}\frac{d^2\phi}{dz^2} + K_t\,\phi \qquad (10.7)$$

Substituting Eq.(10.3) in Eq.(10.7) the governing non-linear differential equation can be obtained as

$$EC_w \frac{d^4\phi}{dz^4} - 6EF\left(\frac{d\phi}{dz}\right)^2 \frac{d^2\phi}{dz^2} - \left(GC_s - \frac{PI_p}{A}\right)\frac{d^2\phi}{dz^2} + K_t\,\phi = 0 \qquad (10.8)$$

The boundary conditions associate with this problem are as follows:

(a) Simply supported end:

$$\phi = 0 \quad \text{and} \quad \frac{d^2\phi}{dz^2} = 0 \qquad (10.9)$$

(b) Clamped end:

$$\phi = 0 \quad \text{and} \quad \frac{d\phi}{dz} = 0 \qquad (10.10)$$

(c) Free end:

$$\frac{d^2\phi}{dz^2} = 0$$

and

$$EC_w \frac{d^3\phi}{dz^3} - 2EF\left(\frac{d\phi}{dz}\right)^3 - \left(GC_s - \frac{PI_p}{A}\right)\frac{d\phi}{dz} = 0 \qquad (10.11)$$

The general solution of Eq.(10.8) can be obtained by numerical methods using computer techniques. However, for the purpose of this thesis, approximate solutions are obtained for simply supported and clamped beams using Galerkin's method.

10.3. SIMPLY SUPPORTED BEAM:

For a beam simply supported at both ends, the boundary conditions are:

$$\phi = 0 \text{ and } \phi'' = 0 \text{ at } Z = 0 \qquad (10.12)$$

and

$$\phi = 0 \text{ and } \phi'' = 0 \text{ at } Z = 1 \qquad (10.13)$$

where primes denote differentiation with respect to the dimensionless length $Z = z/L$.

Eq.(10.8) can be written in non-dimensional form as:

$$\phi^{iv} - 6\overset{*}{\delta}(\phi')^2 \phi'' - (K^2 - \Delta^2)\phi'' + 4\gamma^2\phi = 0 \qquad (10.14)$$

where

$$\overset{*}{\delta} = F/C_w \qquad (10.15)$$

To solve Eq.(10.14) by Galerkin's method, the angle of twist $\phi(Z)$ is assumed to be of the form

$$\phi(Z) = \overset{*}{\beta}\chi(Z) \qquad (10.16)$$

where $\overset{*}{\beta}$ is the torsional amplitude and χ is a function of Z. Since χ will be an approximate function assumed to satisfy the boundary conditions, by substituting Eq.(10.16) in Eq.(10.14), an error ϵ^* will be obtained as:

$$\epsilon^* = \beta^* \left[\chi^{iv} - 6\, \beta^{*2}\, \overset{*}{\delta} (\chi')^2 \chi'' - (K^2 - \Delta^2) \chi'' + 4\gamma^2 \chi \right] \qquad (10.17)$$

For minimizing the error ϵ^*, the Galerkin's Integral (79) is

$$\int_0^1 \epsilon^* \chi \, dZ = 0 \qquad (10.18)$$

To satisfy the boundary conditions, Eqs.(10.12) and (10.13), we assume

$$\chi(Z) = \sin \pi Z \qquad (10.19)$$

Substituting Eqs.(10.17) and (10.19) into Eq.(10.18), we obtain the expression for the torsional post-buckling load for a simply supported beam as:

$$\Delta^{*2}_{cr} = K^2 + \pi^2 + 4\gamma^2/\pi^2 + (3/2)\, \pi^2\, \overset{*}{\delta}\, \beta^{*2} \qquad (10.20)$$

The corresponding linear torsional buckling load is given by (See (Eq.2.55)

$$\Delta^2_{cr} = K^2 + \pi^2 + 4\gamma^2/\pi^2 \qquad (10.21)$$

Hence, the ratio of the non-linear buckling load to linear buckling load is given by

$$\frac{P^*}{P_{cr}} = \frac{\Delta^{*2}_{cr}}{\Delta^2_{cr}} = 1 + \frac{(3/2)\pi^4\, \overset{*}{\delta}\, \beta^{*2}}{\left[\pi^2(K^2 + \pi^2) + 4\gamma^2\right]} \qquad (10.22)$$

In the absence of elastic foundation, i.e., $\gamma = 0$, Eq.(10.22) reduces to

$$\frac{P^*}{P_{cr}} = \frac{\Delta^{*2}_{cr}}{\Delta^2_{cr}} = \left[1 + \frac{3\pi^2 \overset{*}{\delta} \beta^{*2}}{2(K^2+\pi^2)}\right] \tag{10.23}$$

10.4. <u>CLAMPED BEAM</u>:

The boundary conditions for a beam clamped at both the ends are:

$$\phi = 0 \quad \text{and} \quad \phi' = 0 \quad \text{at } Z = 0 \tag{10.24}$$

and

$$\phi = 0 \quad \text{and} \quad \phi' = 0 \quad \text{at } Z = 1 \tag{10.25}$$

To satisfy the above conditions, the function $\chi(Z)$ can be assumed as:

$$\chi(Z) = \beta^*(1-\cos 2\pi Z) \tag{10.26}$$

Substituting Eqs.(10.17) and (10.26) into Eq.(10.18) we obtain the expression for the torsional post-buckling load for a clamped beam as:

$$\Delta^{*2}_{cr} = K^2 + 4\pi^2 + 3\gamma^2/\pi^2 + 6\,\pi^2 \overset{*}{\delta} \beta^{*2} \tag{10.27}$$

The corresponding linear torsional buckling load for a clamped beam is (See Eq.2.74)

$$\Delta^2_{cr} = K^2 + 4\pi^2 + 3\gamma^2/\pi^2 \tag{10.28}$$

Hence, the ratio of the non-linear buckling load to linear buckling load is given by

$$\frac{P^{*}}{P_{cr}} = \frac{\Delta_{cr}^{*2}}{\Delta_{cr}^{2}} = \left\{ 1 + \frac{6\pi^{4}\delta^{*}\beta^{*2}}{\left[\pi^{2}(K^{2}+4\pi^{2}) + 3\nu^{2}\right]} \right\} \tag{10.29}$$

In the absence of elastic foundation, ie., $\nu = 0$, Eq.(10.29) reduces to

$$\frac{P^{*}}{P_{cr}} = \frac{\Delta_{cr}^{*2}}{\Delta_{cr}^{2}} = \left[1 + \frac{6\pi^{2}\delta^{*}\beta^{*2}}{K^{2} + 4\pi^{2}} \right] \tag{10.30}$$